Algebraic Geometry: Spaces of Physical Possibility

Part III of *A Math→Physics Representation Library*

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Abstract

This is Part III of the modular series *A Math→Physics Representation Library*, which formalizes each mathematical domain as a physical-representation module and composes the modules hierarchically. Part I introduced the *representation stack* and the realization pipeline $\mathrm{Obs}_\alpha(M) = \mathrm{Obs}(\mathrm{Real}_\alpha(\Phi(M)))$; Part II gave observables an internal *coalgebraic anatomy* through motives, periods, and the Goncharov coproduct. Here we supply the geometric *stage* on which those motives and periods live and vary: the theory of schemes, moduli spaces and stacks, variations of Hodge structure, and positive geometries. Our organizing thesis is that *algebraic geometry is the geometry of physical possibility*: an affine variety is a classical solution space, a scheme is a solution space enriched with infinitesimal structure, a moduli space is a space of physically distinct configurations, and a moduli *stack* is such a space with its gauge automorphisms remembered rather than quotiented away. We work throughout with the epistemic status discipline of Part I, labelling every dictionary entry **[S]** (standard), **[H]** (heuristic), or **[P]** (speculative). We prove four theorems that upgrade Part II's per-object realization pipeline to a family-indexed one: (T1) Gauss–Manin flat transport realizes the Part II period functor over an entire moduli family, with amplitudes constrained by the associated Picard–Fuchs system; (T2) a moduli stack $[X/G]$ is a strictly finer representation object than its coarse space exactly when stabilizers are nontrivial, giving a rigorous algebro-geometric form of Part I's "stacks retain gauge information" principle; (T3) the recursive residue structure of a positive geometry reproduces a coalgebra-like decomposition tree conjecturally isomorphic to Part II's motivic coaction tree (flagged **[H]**); and (T4) the derived critical locus formalizes the on-shell observable algebra R/I at the Batalin–Vilkovisky level, upgrading a heuristic dictionary entry to a status-**[S]** statement via shifted symplectic geometry. We accompany the paper with executable Haskell encodings of schemes-as-functors-of-points, quotient stacks, variations of Hodge structure, and positive geometries, together with a best-effort Lean 4 sketch of the same

type signatures. Part III thereby sets the geometric foundation on which Part IV (algebraic topology: conserved global information) will build.

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1 Introduction

1.1 The series and its modular ladder

The present paper is the third module of a seven-paper program, *A Math→Physics Representation Library*. The program takes as its point of departure the observation that many physical quantities are not merely *described by* mathematics but are literally *obtained as realizations* of structured mathematical information [1]. Rather than assert one universal functor from all of mathematics to all of physics (an object we argue in Section 12.1 is unlikely to be well defined), the program is deliberately *modular*: each mathematical domain is formalized as its own physical-representation module, each module is defensible on its own terms, and the modules compose hierarchically so that new structure and new emergent physical content appear at each level.

The ladder proceeds as follows.

- **Part I (Foundations).** Introduces the category Dom of mathematical domains, the categories Math_d of structures in each domain, the target Phys of physical representations, the notion of a *representation entry* $E = (M, P, \tau, \sigma)$, the *representation prestack* and *representation stack*, the four foundational axioms (Realization, Equivalence, Locality/descent, Decomposition), and the epistemic status system **[S]**/**[H]**/**[P]**. Its central formula is the *realization pipeline*
$$\mathrm{Obs}_\alpha(M) \;=\; \mathrm{Obs}\big(\mathrm{Real}_\alpha(\Phi(M))\big). \tag{1}$$
- **Part II (Motives, periods, amplitudes).** Makes the abstract realization channels Real_α of Part I concrete: Betti, de Rham, Hodge, and étale realizations of motives; the period datum $\Pi = (X, D, \omega, \Gamma)$ with $\mathrm{per}(\Pi) = \int_\Gamma \omega$; the motivic amplitude object $\mathcal{A}^{\mathfrak{m}}$ with $\mathrm{per}(\mathcal{A}^{\mathfrak{m}}) = \mathcal{A}$; and the coalgebraic anatomy encoded by the coproduct Δ and cobracket δ. Observables acquire an internal *decomposability* into primitive informational pieces.

- **Part III (this paper).** Supplies the geometric *stage*. Part II's motives and periods do not live in a vacuum: a period datum requires X to be an actual algebraic or analytic space, and a *family* of amplitudes over kinematic space is a variation of Hodge structure over a moduli space. We formalize schemes, moduli spaces and stacks, Hodge theory and Gauss–Manin transport, and positive geometries, and we show that Part II's per-object pipeline extends to a *family* pipeline fibered over the geometry.

- **Parts IV–VII.** Algebraic topology (conserved global information), category theory and homotopy type theory (composition and identity), sheaves/stacks/gauge redundancy/quantum high availability, and finally a synthesis paper recomposing the ladder. Part III sets up Part IV by producing the varieties and moduli whose topological invariants Part IV will study.

1.2 Thesis of Part III

Our thesis, following the primary source's algebraic-geometry appendix, is compact:

Algebraic geometry is the geometry of physical possibility.

The word "possibility" is meant in a precise, layered sense that the machinery of algebraic geometry makes available:

1. **Solutions as points.** The set of states satisfying a system of polynomial constraints is an *affine variety* $V(\mathfrak{a})$; its coordinate ring $R/\mathfrak{a}$ is the algebra of on-shell observables. **[S]** as algebra; the reading "measurements generate the space of possibilities" is **[H]**.

2. **Infinitesimal possibility.** Passing from varieties to *schemes* $\operatorname{Spec} R$ retains nilpotent (infinitesimal, off-shell, ghost-like) directions that varieties discard; possibility acquires a tangent/deformation structure.

3. **Families of possibility.** A *moduli space* parametrizes physically distinct configurations; the space itself, its singularities, and its compactification become physical objects (vacua, thresholds, asymptotic sectors).

4. **Possibility modulo redundancy.** A *moduli stack* $[X/G]$ remembers the gauge automorphisms (stabilizers) of each configuration, distinguishing genuine physical states from redundant descriptions, which is the geometric incarnation of Part I's Equivalence axiom.

5. **Possibility varying analytically.** A *variation of Hodge structure* equips a family with a flat Gauss–Manin connection whose flat sections are the Betti cycles; pairing them against the (non-flat) holomorphic form gives the amplitudes across kinematic space, which obey the Picard–Fuchs differential equations.

6. **Possibility with a boundary.** A *positive geometry* carries a canonical form whose boundary residues recursively encode lower-dimensional geometries, a geometric model of physical factorization and singularity structure.

1.3 Contributions

1. A self-contained formalization (Sections 2 and 3) of the algebraic-geometry layer of the representation library: schemes via the functor of points, ideals as constraints, the on-shell algebra $R/\mathfrak{a}$, and the full **[S]**/**[H]**/**[P]**-labelled dictionary of Section 4, reproduced and organized from the primary source.

2. Four theorems with proofs (Section 8), each of which extends a Part II construction over a geometric family: Gauss–Manin family realization (Theorem 8.1), stack-vs-coarse strictness (Theorem 8.3), positive-geometry residue/coaction correspondence (Theorem 8.5, status **[H]**), and the derived critical locus as BV on-shell algebra (Theorem 8.7).

3. An explicit family realization pipeline (Section 9) generalizing (1) from a single mathematical object to a moduli-indexed family, together with a status-propagation calculus inherited from Part I.

4. Executable Haskell encodings and a best-effort Lean 4 sketch (Section 11) of schemes-as-functors-of-points, quotient stacks with stabilizer data, variations of Hodge structure with Gauss–Manin connections, and positive geometries with recursive residue trees.

5. An explicit accounting (Section 12) of how Part III respects the program's five standing limitations, and a list of open problems, notably the conjectural identification of positive-geometry residue trees with motivic coaction trees, handed forward to Parts IV–VII.

1.4 Epistemic status discipline

We preserve verbatim the three-level status system of Part I.

Label	Meaning	Canonical example
[S]	standard use in mathematics or mathematical physics	TQFT as a functor $\mathsf{Bord}_n \to \mathsf{Vect}$
[H]	strong heuristic representation dictionary entry	motive as "functional essence"
[P]	speculative philosophical ontology	reality as motivic-categorical realization

Composite labels (**[S/H]**, **[H/P]**) appear where an entry is standard as pure mathematics but heuristic or speculative in its *physical* reading; we preserve these composite labels verbatim. This discipline is not decoration: it is the program's central epistemic-honesty device, and in Section 8 we state precisely which of our four theorems are status-**[S]** (established mathematics applied to a physical reading), which are status-**[H]** (heuristically plausible, original to this program), and which are status-**[P]**.

2 Mathematical Framework

We recall the ambient framework of Parts I–II in the compressed form we need, then specialize to the algebraic-geometry domain $d = \mathsf{AG}$.

2.1 Domains, structures, and representation entries

Definition 2.1 (Representation entry, after Part I)**.** Fix a category Dom of mathematical domains and, for each $d \in \mathsf{Dom}$, a (possibly higher) category Math_d of mathematical structures in d, together with a category Phys of physical representations. A *representation entry* is a quadruple $E = (M, P, \tau, \sigma)$ where $M \in \mathsf{Math}_d$, $P \in \mathsf{Phys}$, τ is a semantic translation datum (a functor, a natural transformation, an equivalence class of models, a physical interpretation map, or a weaker relation), and $\sigma \in \{\mathbf{[S]}, \mathbf{[H]}, \mathbf{[P]}\}$ is the epistemic status label.

For this paper $d = \mathsf{AG}$ and $\mathsf{Math}_{\mathsf{AG}}$ is (a chosen full subcategory of) the category of schemes, algebraic spaces, and stacks over a fixed base field $\daleth$, together with the auxiliary categories of Hodge structures and positive geometries introduced below.

Definition 2.2 (Status monoid, after Part I)**.** Order the labels by $\mathbf{[S]} < \mathbf{[H]} < \mathbf{[P]}$ and equip $\{\mathbf{[S]}, \mathbf{[H]}, \mathbf{[P]}\}$ with the binary operation $\sigma_1 \wedge \sigma_2 := \max(\sigma_1, \sigma_2)$ (the *less reliable* of the two). This is a commutative monoid with unit $\mathbf{[S]}$. For composite labels we take the join componentwise; e.g. $\mathbf{[S/H]} \wedge \mathbf{[S]} = \mathbf{[S/H]}$ and $\mathbf{[S/H]} \wedge \mathbf{[H]} = \mathbf{[H]}$.

Remark 2.3. Theorem 2.2 operationalizes Part I's status system into a calculus: a chain of translations is only as reliable as its weakest link, so the status of a composite entry is the $\wedge$-join of the component statuses. We use this repeatedly to certify that no theorem in Section 8 silently launders a heuristic step into a standard conclusion.

2.2 The realization pipeline

Recall the universal formula (1). In Part II the channels α were the cohomological realizations $\{\text{Betti}, \text{de Rham}, \text{Hodge}, \text{étale}\}$ and the abstract information functor was $\Phi = \text{Mot}$, sending a variety to its motive. The realization pipeline for a single variety X is the chain

$$X \xmapsto{\text{Mot}} \text{Mot}(X) \xmapsto{\text{Real}_\alpha} \text{Real}_\alpha(\text{Mot}(X)) \xmapsto{\text{Obs}} \text{observable.} \tag{2}$$

Part III's central move is to let X *vary*: we replace the single object X by a family $\pi\colon \mathcal{X} \to S$ over a base scheme (moduli space) S and ask how (2) behaves as we move in S. The answer, made precise in Theorem 8.1, is that the realization assembles into a variation of Hodge structure over S: the Betti cycles are the flat sections of the Gauss–Manin connection, and the observable is their pairing with the (non-flat) holomorphic form, a period function whose variation is governed by ∇_{GM} and which satisfies the Picard–Fuchs equations.

2.3 The master diagram, fibered over a base

Figure 1 exhibits the family realization pipeline as a commuting diagram. The top row is Part I's master diagram for a single object; the bottom row is its family version, and the vertical arrows are the fibers of the family over a point $s \in S$.

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, "\mathrm{Mot}"] \arrow[d, "\pi"'] & \underline{\mathrm{Mot}}(\mathcal{X}/S) \arrow[r, "\mathrm{Real}_\alpha"] \arrow[d] & \mathcal{H}_\alpha \arrow[r, "\mathrm{Obs}=\int_\gamma"] \arrow[d, "\text{fiber}"] & \text{period } \Pi(s) \arrow[d] \\
S \arrow[r, equal] & S \arrow[r, equal] & S \arrow[r, "\nabla_{\mathrm{GM}}"'] & \text{(Picard–Fuchs)}
\end{tikzcd}
$$

Figure 1: The family realization pipeline of Part III. Over each $s \in S$ the fiber recovers Part II's single-object chain (2); globally the realization Real_α produces a variation of Hodge structure $\mathcal{H}_\alpha$ carrying the Gauss–Manin connection ∇_{GM}. The observable $\mathrm{Obs} = \int_\gamma$ pairs the (non-flat) holomorphic section of $\mathcal{H}_\alpha$ against the flat Betti cycles $\gamma(s)$, yielding the period functions $\Pi(s)$ — the amplitudes over kinematic space — which solve the Picard–Fuchs system (Theorem 8.1). The flat sections of ∇_{GM} are the Betti cycles, not the periods.

3 Schemes as Spaces of Physical Possibility

3.1 From constraints to varieties

Definition 3.1 (Affine variety, on-shell algebra)**.** Let $\daleth$ be a field and $R = \daleth[x_1, \dots, x_n]$. For an ideal $\mathfrak{a} \subseteq R$ generated by polynomials $f_1, \dots, f_r$ (the *constraint equations*), the *affine algebraic set* is the zero locus

$$V(\mathfrak{a}) \;=\; \{\, a \in \daleth^n : f_i(a) = 0,\ i = 1, \dots, r \,\},$$

and the *on-shell observable algebra* is the quotient ring $R/\mathfrak{a}$.

Remark 3.2 (Terminology). In the strict algebro-geometric sense an *affine variety* is an *irreducible* affine algebraic set, i.e. $\mathfrak{a}$ is a prime ideal. Throughout this paper we use "variety" in the looser, physically natural sense of an arbitrary solution space $V(\mathfrak{a})$ (an affine algebraic set), since a physical constraint system need not cut out an irreducible locus. Where irreducibility is actually used (e.g. for a connected smooth family in Theorem 8.1) we say so explicitly.

Physically, $V(\mathfrak{a})$ is the space of classical states satisfying the imposed equations of motion or conservation constraints, and $R/\mathfrak{a}$ is the algebra of functions (observables) modulo those constraints: an observable that differs from another by an element of $\mathfrak{a}$ takes the same value on every physical state, hence *is* the same observable on-shell.

Example 3.3 (On-shell observable algebra, primary-source Example, **[S]**/**[H]**)**.** Let R be an algebra of fields and coordinates and let $\mathfrak{a}$ be the ideal generated by the equations of motion or constraints. Then $R/\mathfrak{a}$ is the natural algebraic representation of observables modulo the imposed physical equations. *Status:* **[S]** as pure algebra; the ontological reading "measurements generate the space of physical possibility" is **[H]**.

Example 3.4 (Free particle constraint surface)**.** Take $R = \mathbb{R}[q, p, E]$ and $\mathfrak{a} = (E - \frac{1}{2m}p^2)$. Then $V(\mathfrak{a}) \subseteq \mathbb{R}^3$ is the paraboloid of on-shell energy-momentum data, and $R/\mathfrak{a} \cong \mathbb{R}[q, p]$: the energy E is not an independent observable on-shell but a function of p. This is the algebraic content of "imposing the mass-shell condition."

3.2 Schemes and infinitesimal possibility

Varieties see only reduced structure: they cannot distinguish $V(x)$ from $V(x^2)$, both being the single point $\{0\}$. Physics, however, frequently needs the infinitesimal or off-shell data that nilpotents encode. Schemes retain this data.

Definition 3.5 (Affine scheme)**.** For a commutative ring R, the *affine scheme* $\operatorname{Spec} R$ is the set of prime ideals of R equipped with the Zariski topology (closed sets $V(\mathfrak{a}) = \{\mathfrak{p} : \mathfrak{a} \subseteq \mathfrak{p}\}$) and the structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ whose stalk at $\mathfrak{p}$ is the localization $R_{\mathfrak{p}}$.

Definition 3.6 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to an affine scheme: every point has an open neighbourhood U with $(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some commutative ring R (see Hartshorne [2], Ch. II).

Remark 3.7 (Functor of points). Equivalently, by Yoneda, a scheme X is determined by its *functor of points* $h_X : \mathsf{CRing} \to \mathsf{Set}$, $R \mapsto \operatorname{Hom}_{\mathsf{Sch}}(\operatorname{Spec} R, X)$; an R-point of X is a $\daleth$-algebra homomorphism from the coordinate ring of X to R. This viewpoint is the one we encode in Haskell and Lean in Section 11: a scheme becomes a rule assigning to each ring of "test parameters" the set of configurations parametrized by that ring. The nilpotent directions are visible precisely because R is allowed to be nonreduced (e.g. dual numbers $\daleth[\varepsilon]/(\varepsilon^2)$, whose points detect tangent vectors).

Example 3.8 (Nilpotents as virtual directions, **[H]**)**.** The scheme $\operatorname{Spec}\big(\daleth[x]/(x^2)\big)$ has one point but a nonzero tangent space: its $\daleth[\varepsilon]/(\varepsilon^2)$-points are $\{0, \varepsilon\}$, detecting a first-order deformation invisible to the underlying variety. In the physical dictionary a nilpotent element is an *infinitesimal thickening* — a virtual, off-shell, or ghost-like direction (status **[H]**). This is the algebraic seed of the Batalin–Vilkovisky ghost directions we recover rigorously in Theorem 8.7.

3.3 Projective varieties and scale invariance

Definition 3.9 (Projective variety)**.** For a homogeneous ideal $\mathfrak{a} \subseteq \daleth[x_0, \dots, x_n]$, the *projective variety* $V_+(\mathfrak{a}) = \operatorname{Proj}\big(\daleth[x_0, \dots, x_n]/\mathfrak{a}\big) \subseteq \mathbb{P}^n_{\daleth}$ is the zero locus of $\mathfrak{a}$ in projective space, i.e. the space of solutions taken modulo the rescaling $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$, $\lambda \in \daleth^{\times}$.

Physically a projective variety is a *scale-invariant configuration space*: physical data considered modulo an overall rescaling. Cross-ratios and other conformal invariants live naturally on projective and Grassmannian geometries; this is the setting in which positive geometries (Section 7) are formulated.

4 The Algebraic Geometry Dictionary

We reproduce and organize the algebraic-geometry library of the primary source (its Appendix C), preserving every status label. The dictionary is the data of a family of representation entries (Theorem 2.1) in the domain $d = \mathsf{AG}$; the theorems of Section 8 pick out the load-bearing rows and prove precise statements about them.

Table 1: Algebraic geometry library (part 1 of 2): schemes, bundles, quotients. Status labels preserved verbatim from the primary source.

Algebraic geometry	**Physical representation**	**Semantic meaning**	**Status**
Affine variety	Classical solution space	Allowed states satisfying polynomial constraints	**[S]**
Projective variety	Scale-invariant configuration space	Physical data modulo rescaling	**[S]**
Scheme	Generalized solution space	Includes infinitesimal/nilpotent structure	**[S/H]**
Spec R	Space encoded by observables	Observables determine geometry	**[S/H]**
Coordinate ring	Algebra of observables/functions	Measurements generate space	**[S]**
Ideal $\mathfrak{a}$	Constraint equations	Equations of motion/conservation constraints	**[S/H]**
Quotient ring $R/\mathfrak{a}$	On-shell observable algebra	Functions modulo physical constraints	**[S]**
Nilpotent element	Infinitesimal thickening	Virtual/off-shell/ghost-like direction	**[H]**
Sheaf	Locally defined data	Local fields glued into global fields	**[S]**
Stalk	Local data at a point	Infinitesimal local physical information	**[S/H]**
Čech cocycle	Gluing rule	Gauge transition function	**[S]**
Sheaf cohomology	Obstruction to global gluing	Global anomaly or obstruction	**[S/H]**
Line bundle	Phase/gauge degree over space	$U(1)$-type local phase structure	**[S]**
Vector bundle	Internal degrees over spacetime	Matter or gauge field carrier	**[S]**
Principal G-bundle	Gauge configuration	Local symmetry fiber at each point	**[S]**

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Algebraic geometry	**Physical representation**	**Semantic meaning**	**Status**
Connection	Gauge potential	Rule for parallel transport	**[S]**
Curvature	Field strength	Failure of transport to commute	**[S]**
Divisor	Codimension-one locus	Defect, pole, boundary, charge surface	**[H]**
Blowup	Resolution of singularity	Regularization/zooming into divergence	**[S/H]**
Exceptional divisor	New boundary after resolution	Counterterm or hidden boundary sector	**[H]**
Compactification	Add boundary at infinity	Include asymptotic states/IR sectors	**[S/H]**
Moduli space	Space of inequivalent structures	Physically distinct vacua/solutions	**[S]**
Moduli stack	Moduli with automorphisms retained	Gauge quotient preserving stabilizers	**[S]**
Quotient stack $[X/G]$	Gauge quotient	Physical states modulo symmetry, with isotropy	**[S]**

Table 2: Algebraic geometry library (part 2 of 2): derived, Hodge-theoretic, and positive-geometric entries. Status labels preserved verbatim from the primary source.

Algebraic geometry	**Physical representation**	**Semantic meaning**	**Status**
Derived scheme/stack	Equations plus higher constraints	BV/BRST-style field space with ghosts	**[S/H]**
Derived critical locus	Critical points with obstruction data	Perturbative field theory around action	**[S/H]**
Tangent complex	Linearized deformation data	Fluctuations around a solution	**[S]**
Cotangent complex	Moment/covector deformation data	Phase-space infinitesimals	**[S/H]**
Symplectic variety	Geometric phase space	Classical mechanics structure	**[S]**
Poisson variety	Bracketed observable space	Classical observable algebra	**[S]**
Calabi–Yau variety	Special geometry/compactification	String background or period source	**[S]**
Elliptic curve	Torus-like period geometry	Elliptic Feynman integral sector	**[S]**

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Table 2 – *continued from previous page*

Algebraic geometry	**Physical representation**	**Semantic meaning**	**Status**
Algebraic curve	Spectral curve	Dispersion/integrability data	**[S]**
Jacobian	Torus of line bundles	Abelianized phase/integrable variables	**[S]**
Picard group	Line-bundle classes	Phase sectors or charge classes	**[S/H]**
Hodge structure	Cohomological decomposition	Analytic-topological complexity	**[S]**
Mixed Hodge structure	Layered singular/relative cohomology	Amplitudes with boundaries/degenerations	**[S]**
Variation of Hodge str.	Periods varying over parameters	Amplitudes over kinematic space	**[S]**
Gauss–Manin connection	Transport of cohomology over moduli	Differential equations for amplitudes	**[S]**
Picard–Fuchs equation	Period differential equation	Equation satisfied by integral/amplitude	**[S]**
Riemann–Hilbert corr.	ODEs to monodromy	Dynamics to analytic-continuation data	**[S]**
Monodromy	Transport around singularity	Branch cuts or analytic continuation	**[S]**
Discriminant locus	Degenerate parameter values	Thresholds or singular kinematics	**[S]**
Positive geometry	Region with canonical form	Geometry encoding scattering amplitude	**[S]**
Canonical form	Distinguished differential form	Amplitude or integrand form	**[S]**
Amplituhedron	Positive geometry for amplitudes	Scattering encoded geometrically	**[S]**
Positive Grassmannian	Positive cell geometry	On-shell amplitude combinatorics	**[S]**
Cluster algebra	Mutating coordinate system	Different amplitude charts/channels	**[S/H]**
Tropical geometry	Degeneration or asymptotic geometry	Classical, graph, or boundary regime	**[S/H]**
Toric variety	Combinatorial algebraic geometry	Phase space from polytope data	**[S]**
Newton polytope	Monomial support geometry	Semiclassical/dominant-balance structure	**[H]**

Remark 4.1 (Reading the dictionary through the status monoid). The dictionary is not a flat list: it is stratified by Theorem 2.2. The purely mathematical content of every row is **[S]** (these are standard objects of algebraic geometry); the composite labels record that the *physical* reading may drop to **[H]**. For instance, "ideal = constraint equations" is **[S/H]** because the algebra is standard but the claim that *every* physical constraint is polynomial is heuristic. When we compose dictionary entries along the realization pipeline, Theorem 2.2 tells us the composite status; e.g. realizing an amplitude via a variation of Hodge structure (**[S]**) over a moduli *space* (**[S]**) is **[S]**, but reading the resulting object as "the" cosmic-Galois-covariant amplitude (a Part II **[H/P]** entry) forces the composite to **[H/P]**.

5 Moduli, Stacks, and Gauge-Correct Possibility

5.1 Moduli functors and stacks

The space of "physically distinct configurations" of a given type is a *moduli space*. We phrase it functorially, which is both the mathematically robust formulation and the one matching our functor-of-points encoding (Theorem 3.7).

Definition 5.1 (Moduli functor, moduli stack)**.** A *moduli functor* for a class of objects with a notion of family is a functor $\mathcal{M}\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ sending a test scheme T to the set of families over T modulo isomorphism. If instead one records the isomorphisms (does not quotient by them), one obtains a *moduli stack* $\underline{\mathcal{M}}\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Gpd}$, valued in groupoids. The moduli functor is *represented* by a scheme (resp. the stack by an algebraic stack) when it satisfies effective descent and admits a universal family.

Definition 5.2 (Quotient stack)**.** Let an algebraic group G act on a scheme X. The *quotient stack* $[X/G]$ is the stackification of the action groupoid $X \times G \rightrightarrows X$ (source and target maps $(x, g) \mapsto x$ and $(x, g) \mapsto g \cdot x$). Its T-points form the groupoid of principal G-bundles $P \to T$ together with a G-equivariant map $P \to X$. The *coarse space* X/G is the scheme (or algebraic space) that best approximates the stack: it is characterized by the universal property of co-representing $[X/G]$, i.e. it receives a canonical map $[X/G] \to X/G$ that is universal among maps from $[X/G]$ to schemes. For a reductive G acting on an affine $X = \operatorname{Spec} A$ it is the GIT quotient $X/\!/G = \operatorname{Spec}(A^G)$ of invariants. We stress that X/G is *not* in general the sheafification of the orbit presheaf $T \mapsto X(T)/G(T)$: when stabilizers are nontrivial that sheaf need not be representable (e.g. for $\mathbb{G}_m \curvearrowright \mathbb{A}^1$ by scaling the invariant quotient is a single point, whereas the fppf orbit sheaf has two points).

The essential invariant retained by the stack but forgotten by the coarse space is the automorphism (stabilizer) group of each point.

Proposition 5.3 (Stabilizer is the automorphism group of a point in the stack)**.** *For $x \in X$, the automorphism group of the corresponding object of the groupoid $[X/G](\operatorname{Spec} \urcorner)$ is canonically isomorphic to the stabilizer $\operatorname{Stab}_G(x) = \{g \in G : g \cdot x = x\}$:*

$$\operatorname{Aut}_{[X/G]}(x) \;\cong\; \operatorname{Stab}_G(x).$$

Proof. By Theorem 5.2, morphisms $x \to x'$ in the action groupoid are group elements $g \in G$ with $g \cdot x = x'$; automorphisms are therefore $g \in G$ with $g \cdot x = x$, i.e. exactly $\operatorname{Stab}_G(x)$.

Stackification is a 2-categorical localization that preserves automorphism groups of objects (it only adds descent data and formal inverses to already-invertible 1-morphisms), so the automorphism group is unchanged in $[X/G]$. $\square$

Theorem 5.3 is the algebraic-geometry incarnation of the master formula $\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x)$ recorded across the whole program. It is the reason a moduli stack is "gauge-correct": it distinguishes a configuration with no gauge automorphisms from one stabilized by a residual gauge subgroup, even when the two have the same image in the coarse space.

5.2 Gauge redundancy, formally

Definition 5.4 (Gauge redundancy, after Part I/VI)**.** A *gauge redundancy* on a configuration space X is the action of a group G whose orbits are to be regarded as single physical states: x and $g \cdot x$ describe *the same* physics. The physically correct state space is the quotient stack $[X/G]$, not merely the set of orbits X/G, whenever stabilizers carry physical information (anomalies, quantization conditions, or measure factors).

Remark 5.5 (Gauge redundancy is not physical symmetry). We reiterate the program's Limitation 5: a gauge redundancy identifies *different descriptions of the same* physical content, whereas a physical symmetry maps one physical state to a *genuinely different* one. The electromagnetic potential $A \mapsto A + d\lambda$ with field strength $F = dA$ (and $d^2 = 0$, so F is unchanged) is the paradigm: A and $A + d\lambda$ are gauge-equivalent descriptions of one physical field configuration. Algebro-geometrically this is the statement that the physical object is a point of a *stack* of connections modulo gauge, and its automorphisms are the residual (constant) gauge transformations.

6 Hodge Theory and Kinematic-Space Amplitudes

We now formalize the machinery that lets Part II's periods *vary* over a family, which is the technical heart of Theorem 8.1.

Definition 6.1 (Hodge structure)**.** A (pure) *Hodge structure* of weight w on a finite-dimensional $\mathbb{Q}$-vector space V is a decomposition $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=w} V^{p,q}$ with $\overline{V^{p,q}} = V^{q,p}$. Equivalently it is a decreasing *Hodge filtration* $F^\bullet$ with $V_{\mathbb{C}} = F^p \oplus \overline{F^{w-p+1}}$. A *mixed* Hodge structure adds an increasing weight filtration $W_\bullet$ with pure Hodge structures on the graded pieces (Deligne).

Definition 6.2 (Variation of Hodge structure)**.** A *variation of Hodge structure* (VHS) over a complex base S is a local system $\mathbb{V}$ of $\mathbb{Q}$-vector spaces on S together with a holomorphic Hodge filtration $F^\bullet \subset \mathbb{V} \otimes \mathcal{O}_S$ such that

1. each fiber $(\mathbb{V}_s, F^\bullet_s)$ is a Hodge structure, and
2. **Griffiths transversality** holds: $\nabla_{\mathrm{GM}} F^p \subseteq F^{p-1} \otimes \Omega^1_S$, where ∇_{GM} is the flat *Gauss–Manin connection* determined by the local system.

Definition 6.3 (Period map and Picard–Fuchs system)**.** Let $\pi\colon \mathcal{X} \to S$ be a smooth projective family and ω_s a family of holomorphic forms with locally constant integration cycles $\gamma(s) \in H_\bullet(\mathcal{X}_s)$. The *period* is $\Pi(s) = \int_{\gamma(s)} \omega_s$. Because the cycles $\gamma(s)$ are flat for ∇_{GM} (they are the flat sections) while $[\omega_s]$ is not, the period Π is a non-constant function satisfying a linear ODE system — the *Picard–Fuchs system* — obtained by expressing the high-order Gauss–Manin derivatives $[\omega_s], \nabla^{\mathrm{GM}}_\partial[\omega_s], \ldots, (\nabla^{\mathrm{GM}}_\partial)^r[\omega_s]$ in the (finite-rank, rank-r) de Rham cohomology basis to produce a linear dependence and pairing the result with $\gamma(s)$:

$$\sum_{j=0}^{r} c_j(s)\,(\nabla^{\mathrm{GM}}_\partial)^j[\omega_s] = 0 \quad\Longrightarrow\quad \mathcal{L}_{\mathrm{PF}}\,\Pi = 0, \;\; \mathcal{L}_{\mathrm{PF}} = \sum_{j=0}^{r} c_j(s)\,\partial^j,$$

a differential operator of order $\le \operatorname{rank} H^\bullet_{\mathrm{dR}}(\mathcal{X}_s)$. (Note $\nabla_{\mathrm{GM}}\Pi = 0$ does *not* hold: the period is not a flat section; see Theorem 8.1.)

Example 6.4 (Legendre family of elliptic curves, **[S]**)**.** The family $E_\lambda\colon y^2 = x(x-1)(x-\lambda)$ over $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ has periods $\Pi(\lambda) = \oint \frac{dx}{y}$ satisfying the hypergeometric Picard–Fuchs equation

$$\lambda(1-\lambda)\,\Pi'' + (1-2\lambda)\,\Pi' - \tfrac{1}{4}\,\Pi = 0,$$

whose solutions are ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \lambda)$ and its analytic continuation. The discriminant locus $\{0, 1, \infty\}$ is exactly where E_λ degenerates (a node appears); the monodromy of Π around these points is the physical branch-cut/threshold data. This is the simplest nontrivial instance of "amplitudes over kinematic space": the elliptic Feynman-integral sector.

7 Positive Geometries and Geometric Factorization

Definition 7.1 (Positive geometry, after Arkani-Hamed–Bai–Lam [6])**.** A *positive geometry* is a pair $(X, X_{\ge 0})$ of a complex variety X and a real top-dimensional region $X_{\ge 0} \subseteq X(\mathbb{R})$ with boundary components $\partial_i X_{\ge 0}$, together with a unique *canonical form* $\Omega(X_{\ge 0})$ — a meromorphic top form with simple poles exactly along the boundary — satisfying the recursive *residue axiom*: for each boundary component,

$$\operatorname{Res}_{\partial_i}\,\Omega(X_{\ge 0}) \;=\; \Omega(\partial_i X_{\ge 0}),$$

i.e. the residue along a facet is the canonical form of that facet, itself a positive geometry of one lower dimension.

Example 7.2 (Interval and simplex, **[S]**)**.** For the interval $[a, b] \subset \mathbb{P}^1$ the canonical form is $\Omega = \left(\frac{1}{x-a} - \frac{1}{x-b}\right)dx = \frac{(b-a)\,dx}{(x-a)(x-b)}$, with $\operatorname{Res}_{x=a}\Omega = +1 = \Omega(\{a\})$ and $\operatorname{Res}_{x=b}\Omega = -1$. Iterating, the standard simplex Δ^n has canonical form the standard volume form with logarithmic singularities on its facets; each residue is the canonical form of a facet simplex. The recursion terminates at vertices (points), whose canonical form is the number ± 1.

Remark 7.3 (Physical reading, **[S]**). In scattering-amplitude applications (the amplituhedron for planar $\mathcal{N} = 4$ super Yang–Mills, and its cousins) the canonical form *is* the amplitude integrand, and the residue axiom *is* the physical factorization of the amplitude on a singularity: the residue on a boundary component reproduces the product of lower-point amplitudes on the corresponding factorization channel. The recursive tree of residues (Theorem 7.4 below) is thus a geometric model of the full singularity structure of the amplitude.

Definition 7.4 (Residue tree)**.** For a positive geometry $(X, X_{\geq 0})$ the *residue tree* $\mathcal{T}(X_{\geq 0})$ is the rooted tree whose root is decorated by $\Omega(X_{\geq 0})$, whose children are the residue trees of the boundary components $\partial_i X_{\geq 0}$, and whose leaves are the 0-dimensional strata decorated by ± 1.

The residue tree is the object we compare, in Theorem 8.5, with the coaction tree of Part II's motivic amplitude objects.

8 Main Results

We now state and prove the four theorems announced in Section 1. Each extends a Part II construction over a geometric family and each carries an explicit status label.

8.1 T1: Gauss–Manin family realization

Theorem 8.1 (Gauss–Manin transport implements the family realization pipeline; status **[S]**)**.** *Let $\pi\colon \mathcal{X} \to S$ be a smooth projective family of complex varieties over a smooth connected base S (so $\mathcal{X}$, S are varieties over $\mathbb{C}$, or complex manifolds), and let $\mathcal{H}_\alpha = R^k\pi^{\mathrm{an}}_*\underline{\mathbb{Q}}$ be the associated Betti local system on the analytification S^{an} (Betti cohomology is a sheaf in the analytic, not the Zariski, topology), with its Hodge-theoretic realization the variation of Hodge structure of Theorem 6.2 carrying the Gauss–Manin connection ∇_{GM}. (The restriction to the complex/analytic setting is needed for the Betti and Hodge realizations and for Ehresmann's theorem below.) Then:*

1. *The assignment $s \mapsto \mathrm{Real}_{\mathrm{Hodge}}\big(\mathrm{Mot}(\mathcal{X}_s)\big)$ is the fiber of $\mathcal{H}_\alpha$ at s. The Betti cycles $\gamma(s) \in H_k(\mathcal{X}_s;\mathbb{Q})$ are the* flat *(locally constant) sections of the* homology *local system $R_k\pi^{\mathrm{an}}_*\underline{\mathbb{Q}}$, dual to the de Rham cohomology bundle $\mathcal{H}_{\mathrm{dR}}$ carrying ∇_{GM}, while the holomorphic form $[\omega_s] \in F^k$ is in general* not *flat: Griffiths transversality gives $\nabla_{\mathrm{GM}}[\omega_s] \in F^{k-1} \otimes \Omega^1_S$, which is nonzero in general. The period $\Pi(s) = \langle \gamma(s), [\omega_s]\rangle = \int_{\gamma(s)} \omega_s$ is the pairing of the flat cycle with the non-flat form, and its variation is*

$$d\,\Pi \;=\; \int_{\gamma(s)} \nabla_{\mathrm{GM}}[\omega_s],$$

   so Π is generally a non-constant function of s. Choosing a holomorphic frame $e_1, \dots, e_r$ of $\mathcal{H}_{\mathrm{dR}}$ with connection matrix $\nabla_{\mathrm{GM}} e_i = \sum_j A_{ij}\, e_j$, the vector of periods $\Pi_i = \int_\gamma e_i$ satisfies the first-order Gauss–Manin system $d\Pi = A\,\Pi$.

2. *Consequently, eliminating the frame, the scalar period $\Pi = \int_\gamma \omega$ satisfies the Picard–Fuchs system $\mathcal{L}_{\mathrm{PF}}\Pi = 0$ (Theorem 6.3), a differential equation of order at most* $\operatorname{rank} H^k_{\mathrm{dR}}(\mathcal{X}_s)$.

3. Part II's per-object realization pipeline ([2](#)) *is the fiber over each $s \in S$ of a* family *realization pipeline*

$$\mathcal{X} \xrightarrow{\text{Mot}} \underline{\text{Mot}}(\mathcal{X}/S) \xrightarrow{\text{Real}_\alpha} \mathcal{H}_\alpha \xrightarrow{\text{Obs}=\int_{\gamma(\cdot)}} \Pi(s),\ \mathcal{L}_{\text{PF}}\Pi = 0,$$

in which Real_α *produces the VHS* $\mathcal{H}_\alpha$ *(with its holomorphic section* $[\omega_s]$*) and* Obs *pairs that section against the flat Betti cycles to output the period functions* $\Pi(s)$*. Thus "amplitudes over kinematic space" are exactly these period functions — solutions of the Picard–Fuchs system of a VHS over the moduli base* S *— not flat sections of the VHS.*

Proof. (1) The Gauss–Manin connection is the connection on the relative de Rham cohomology bundle $\mathcal{H}_{\text{dR}} = R^k\pi_*\Omega^\bullet_{\mathcal{X}/S}$ characterized by declaring the image of the topological (Betti) local system $R^k\pi_*\underline{\mathbb{Q}}$ to be its flat sections; equivalently it is the unique flat connection whose horizontal sections are the locally constant cohomology classes (Katz–Oda [13]). Dually, the integration cycles $\gamma(s) \in H_k(\mathcal{X}_s;\mathbb{Q})$ form a flat section of the homology local system because $H_k(\mathcal{X}_s)$ is locally constant in s (Ehresmann: π is a proper submersion, hence a locally trivial fiber bundle in the analytic topology); thus $\nabla^\vee_{\text{GM}}\gamma = 0$. By contrast the holomorphic form represents a section $[\omega_s]$ of the Hodge sub-bundle $F^k \subseteq \mathcal{H}_{\text{dR}}$, and Griffiths transversality gives $\nabla_{\text{GM}}[\omega_s] \in F^{k-1} \otimes \Omega^1_S$, which is nonzero in general: $[\omega_s]$ is *not* flat. Differentiating the pairing $\Pi(s) = \langle\gamma(s), [\omega_s]\rangle$ by the Leibniz rule for the flat pairing,

$$d\,\Pi(s) = \underbrace{\langle\nabla^\vee_{\text{GM}}\gamma, [\omega_s]\rangle}_{=0\ \text{since}\ \nabla^\vee_{\text{GM}}\gamma=0} + \langle\gamma, \nabla_{\text{GM}}[\omega_s]\rangle = \int_{\gamma(s)} \nabla_{\text{GM}}[\omega_s],$$

where the first term vanishes precisely because the Betti cycle γ is flat ($\nabla^\vee_{\text{GM}}\gamma = 0$). This is generally nonzero, so Π is a non-constant function of s. Choosing a holomorphic frame $e_1, \ldots, e_r$ of $\mathcal{H}_{\text{dR}}$ with $\nabla_{\text{GM}}e_i = \sum_j A_{ij}e_j$, the vector of periods $\Pi_i = \int_\gamma e_i$ obeys $d\Pi_i = \int_\gamma \nabla_{\text{GM}}e_i = \sum_j A_{ij}\Pi_j$, i.e. the first-order Gauss–Manin system $d\Pi = A\,\Pi$. (In particular the period is emphatically *not* annihilated by ∇_{GM}; the flat sections are the Betti cycles, against which the non-flat form is measured.)

(2) Since $\mathcal{H}^k_{\text{dR}}(\mathcal{X}_s)$ is finite-dimensional of rank $r = \operatorname{rank} H^k_{\text{dR}}(\mathcal{X}_s)$, for a chosen vector field ∂ on S the $r+1$ classes

$$[\omega],\ \nabla^{\text{GM}}_\partial[\omega],\ \ldots,\ (\nabla^{\text{GM}}_\partial)^r[\omega]$$

lie in an r-dimensional space, hence are linearly dependent over the function field of S: $\sum_{j=0}^r c_j(s)\,(\nabla^{\text{GM}}_\partial)^j[\omega] = 0$. Pairing with the flat cycle $\gamma(s)$ and using $\partial^j\Pi = \int_\gamma(\nabla^{\text{GM}}_\partial)^j[\omega]$ (which follows from flatness of γ as in (1)) yields $\sum_j c_j(s)\,\partial^j\Pi = 0$, i.e. $\mathcal{L}_{\text{PF}}\Pi = 0$ with $\mathcal{L}_{\text{PF}} = \sum_j c_j(s)\partial^j$ of order $\le r$.

(3) The functor $\underline{\text{Mot}}$ of relative motives, the Hodge realization Real_α producing $\mathcal{H}_\alpha$, and the observable map $\text{Obs} = \int_{\gamma(\cdot)}$ (pairing the holomorphic section against flat Betti cycles) compose to the displayed family pipeline; restricting to a fiber over s recovers (2) for $X = \mathcal{X}_s$ by construction of the relative constructions as fiberwise ones. Each of the three constructions is standard (Gauss–Manin/VHS theory), so by Theorem 2.2 the composite

is status **[S]**. The identification of the resulting period functions $\Pi(s)$ — solutions of the Picard–Fuchs system — with "amplitudes over kinematic space" is the physical reading and is the standard content of the differential-equation approach to Feynman integrals. □

Corollary 8.2 (Amplitudes obey their Picard–Fuchs equation)**.** *Any physical amplitude that arises as a period of a smooth projective family over kinematic space S is annihilated by a Picard–Fuchs operator of order at most the rank of the relevant de Rham cohomology; in particular the analytic continuation of the amplitude around a discriminant point of S is controlled by the monodromy of $\mathcal{H}_\alpha$ (branch cuts = monodromy,* **[S]***).*

Proof. Immediate from Theorem 8.1(2) and the Riemann–Hilbert correspondence relating the Picard–Fuchs $\mathcal{D}$-module to its monodromy representation. □

Theorem 8.1 is the precise sense in which Part III "fibers" Part II: the per-object period becomes a period *function* on the moduli base, constrained by the Picard–Fuchs system, and the "amplitude over kinematic space" becomes a rigorously constrained object.

8.2 T2: Stacks are strictly finer than coarse spaces

Theorem 8.3 (Moduli stack is strictly finer than the coarse space; status **[S]**)**.** *Let G be an algebraic group acting on a scheme X over a field $\daleth$, and suppose some point $x \in X(\daleth)$ has nontrivial stabilizer* $\mathrm{Stab}_G(x) \neq 1$*. Then:*

1. (No hypotheses beyond the above.) *The canonical map $q\colon [X/G] \to X/G$ to the coarse space is not an isomorphism in any neighbourhood of the image of x: it is not fully faithful there, because it kills the nontrivial automorphism group* $\mathrm{Aut}_{[X/G]}(x) \cong \mathrm{Stab}_G(x)$ *(Theorem 5.3).*

2. (Local structure.) *Assume in addition that G is* smooth reductive*, that* $\mathrm{char}\,\daleth = 0$*, and that x is a smooth point contained in a G-invariant affine open; equivalently, argue after passing to formal completions. Then there is a locally closed* $\mathrm{Stab}_G(x)$*-invariant affine slice $W \ni x$ and an étale (henselian) neighbourhood U of $q(x)$ for which*

$$[X/G] \times_{X/G} U \;\simeq\; \big[\,\mathrm{Spec}(\mathcal{O}^h_{W,x}) \,/\, \mathrm{Stab}_G(x)\,\big],$$

recording exactly the deformation-and-automorphism data that X/G forgets (étale slice / Luna-type local structure). Without the reductivity/characteristic hypotheses the corresponding statement still holds after passing to formal completions at x.

Proof. (1) The coarse space X/G is by construction a *scheme* (or algebraic space): every point of a scheme has trivial automorphism group. The map q sends the object x of $[X/G](\mathrm{Spec}\,\daleth)$ to its image point. On automorphism groups q therefore induces the map $\mathrm{Aut}_{[X/G]}(x) \cong \mathrm{Stab}_G(x) \to \mathrm{Aut}_{X/G}(q(x)) = 1$, which is not injective when $\mathrm{Stab}_G(x) \neq 1$. A morphism of stacks that is not injective on some automorphism group is not fully faithful, hence not an isomorphism, on any open substack containing x.

(2) Under the stated hypotheses (G smooth reductive, $\mathrm{char}\,\daleth = 0$, and x a smooth point in a G-invariant affine open), Luna's étale slice theorem [14] provides a locally closed $\mathrm{Stab}_G(x)$-invariant affine slice $W \ni x$ transverse to the orbit $G \cdot x$. The multiplication map $G \times_{\mathrm{Stab}_G(x)}$

$W \to X$ is étale onto a saturated neighbourhood of the orbit, and passing to stack quotients gives an étale equivalence $[G \times_{\mathrm{Stab}_G(x)} W/G] \simeq [W/\operatorname{Stab}_G(x)]$ over a neighbourhood of x; taking the local (henselian) ring of W at x yields the displayed local model. This is precisely the automorphism-plus-deformation data lost by the coarse space, whose local ring at $q(x)$ is the ring of $\mathrm{Stab}_G(x)$-invariants $\left(\mathcal{O}^h_{W,x}\right)^{\mathrm{Stab}_G(x)}$. When the reductivity or characteristic-zero hypotheses fail, one replaces the étale slice by its formal analogue: the same local model holds after completing at x, working with the $\mathrm{Stab}_G(x)$-action on the formal deformation neighbourhood of x. □

Corollary 8.4 ($\overline{\mathcal{M}}_g$ must be a stack)**.** *The Deligne–Mumford moduli of stable genus-g curves $\overline{\mathcal{M}}_g$ has points (e.g. hyperelliptic curves, or the curve with extra automorphisms) with non-trivial automorphism groups; hence by Theorem 8.3 it is not faithfully represented by its coarse space. Whenever curves with automorphisms contribute physically distinct weightings — as in the string-theoretic path integral over the moduli of worldsheets, where the automorphism group divides the measure — one must integrate over the* stack*, not the coarse space.*

Proof. Deligne–Mumford [4] construct $\overline{\mathcal{M}}_g$ as a smooth proper DM stack; the existence of automorphism-carrying curves is classical (e.g. the genus-2 curve $y^2 = x^5 - x$ has automorphism group of order > 1). Apply Theorem 8.3. □

Theorem 8.3 is the algebro-geometric member of the program's four-fold "stacks retain gauge information" family: Part I posits it informally, Part III proves it geometrically here, Part V will prove a type-theoretic version (groupoid quotient vs. set truncation), and Part VI will prove the descent-theoretic version. All four are the same fact at increasing rigor.

8.3 T3: Positive-geometry residues and the motivic coaction

Theorem 8.5 (Residue tree reproduces a coaction-like decomposition; status **[H]**)**.** *Let $(X, X_{\geq 0})$ be a positive geometry with canonical form Ω and residue tree $\mathcal{T}(X_{\geq 0})$ (Theorem 7.4). Then:*

1. *(Status* **[S]***.) The residue operation defines a coalgebra-like "geometric coproduct"*

$$\Delta_{\mathrm{geo}}\, \Omega(X_{\geq 0}) \;=\; \sum_i \Omega(\partial_i X_{\geq 0}) \otimes \big[\textit{normal data at } \partial_i\big],$$

   and iterating Δ_{geo} reproduces the tree $\mathcal{T}(X_{\geq 0})$; this operation is coassociative up to the ordering of iterated boundaries (the boundary strata form a poset, giving a graded coalgebra structure on the free vector space on strata).

2. *(Status* **[H]***, conjectural for amplituhedron-type geometries.) When Ω is the integrand of a Feynman/scattering amplitude whose period admits a motivic lift $\mathcal{A}^{\mathfrak{m}}$ (Part II), the tree $\mathcal{T}(X_{\geq 0})$ is isomorphic, as a graded rooted tree, to the weight-graded coaction tree of $\mathcal{A}^{\mathfrak{m}}$ under the motivic coproduct $\Delta(\mathcal{A}^{\mathfrak{m}}) = \sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}$; i.e. geometric factorization on boundaries and motivic coaction on periods are the same decomposition.*

Proof. (1) By the residue axiom (Theorem 7.1), $\mathrm{Res}_{\partial_i}\Omega = \Omega(\partial_i X_{\geq 0})$, so the map Δ_{geo} is well defined on the free graded vector space $C_\bullet$ spanned by the (canonical forms of the) boundary strata, graded by codimension. Coassociativity up to poset ordering is the statement that the iterated residue along a codimension-two stratum is independent of the order in which the two facets are approached whenever the strata meet transversally — a standard property of iterated residues of forms with logarithmic (simple-pole) singularities (Leray). Hence $(C_\bullet, \Delta_{\mathrm{geo}})$ is a graded coalgebra and its cogenerated tree is $\mathcal{T}(X_{\geq 0})$. This part is a theorem of positive-geometry theory and is status **[S]**.

(2) This is the conjectural identification and we flag it **[H]**. The evidence is: (a) both sides are weight/codimension-graded rooted trees whose leaves are $\mathbb{Q}$-numbers (residues $= \pm 1$ on one side, weight-0 periods = rationals on the other); (b) for one-loop integrands and for the polygon/associahedron geometries the two trees are known to agree, matching the Goncharov coproduct on the associated polylogarithms; (c) the "cosmic Galois"/coaction principle of Brown and the residue recursion of Arkani-Hamed–Bai–Lam are both governed by the same weight filtration. A general proof would require a functor from the poset of boundary strata of $X_{\geq 0}$ to the comodule of motivic periods intertwining Δ_{geo} with Δ; constructing this functor for all amplituhedron-type geometries is open. We therefore assert (2) as a status-**[H]** candidate theorem, not an established one, and record it in Section 12 as the program's sharpest open problem in this domain. □

Remark 8.6. The honest status bookkeeping of Theorem 8.5 is exactly what the **[S]**/**[H]**/**[P]** discipline is for: part (1) is a genuine theorem, part (2) is a precise conjecture. Collapsing the two would be the kind of overclaim the program is designed to prevent (Limitation 1).

8.4 T4: The derived critical locus as BV on-shell algebra

Theorem 8.7 (Derived critical locus formalizes $R/\mathfrak{a}$ at the BV level; status **[S]**). *Let X be a smooth scheme and $S\colon X \to \mathbb{A}^1$ an action functional (regular function). Let* $\mathrm{Crit}(S) = \mathrm{Spec}\big(\mathcal{O}_X/(dS)\big)$ *be the classical critical locus and* $\mathbf{dCrit}(S)$ *the* derived *critical locus, i.e. the derived zero locus of $dS \in \Gamma(X, \Omega^1_X)$, constructed as the derived intersection of the graph of dS with the zero section inside T^*X:*

$$\mathbf{dCrit}(S) \;=\; X \times^h_{T^*X} X.$$

Then:

1. *The classical truncation is $\pi_0(\mathbf{dCrit}(S)) = \mathrm{Crit}(S)$; in particular $H^0(\mathcal{O}_{\mathbf{dCrit}(S)}) = \mathcal{O}_X/(dS) = R/\mathfrak{a}$ with $\mathfrak{a} = (dS)$ the ideal of equations of motion. Thus the on-shell observable algebra of Theorem 3.3 is the H^0 of the derived object.*

2. $\mathbf{dCrit}(S)$ *carries a canonical (-1)-shifted symplectic structure (Pantev–Toën–Vaquié–Vezzosi), and its structure sheaf, computed via the Koszul resolution of (dS), is the field/antifield sector of the Batalin–Vilkovisky complex: the shifted symplectic form is the BV antibracket, the degree-0 generators are the fields, and the degree-(-1) (and lower) generators are the antifields. No ghosts (positive cohomological degree) appear, because X is a scheme with no gauge symmetry; ghosts are produced only when X is upgraded to an algebraic* stack $[X/G]$ *resolving a gauge redundancy (see Theorem 8.8).*

Consequently the dictionary entry "derived critical locus $\leftrightarrow$ perturbative field theory around the action" is upgraded from **[S/H]** *to* **[S]** *for the field/antifield sector.*

Proof. (1) The section dS of Ω^1_X has classical zero locus $\mathrm{Crit}(S) = V(dS)$; its coordinate ring is $\mathcal{O}_X/(dS)$, which is exactly the on-shell algebra $R/\mathfrak{a}$ of Theorem 3.3 for $\mathfrak{a} = (\partial_i S)$. Derived intersection only refines the vanishing locus by remembering the Koszul syzygies; passing to π_0 (equivalently H^0 of the structure complex) discards the higher Koszul terms and returns the classical quotient. Hence $\pi_0(\mathbf{dCrit}(S)) = \mathrm{Crit}(S)$ and $H^0 = R/\mathfrak{a}$.

(2) Model $\mathbf{dCrit}(S)$ by the Koszul complex $\big(\mathcal{O}_X \otimes \Lambda^\bullet T_X,\, \iota_{dS}\big)$, the Koszul complex contracting with dS; this is the algebra of polyvector fields with differential $\iota_{dS} = \{S, -\}$, i.e. the field/antifield part of the BV complex, with $\Lambda^p T_X$ concentrated in cohomological degree $-p$: degree 0 carries the fields $\mathcal{O}_X$ and degrees $-1, -2, \ldots$ carry the antifields $T_X, \Lambda^2 T_X, \ldots$. Every generator sits in *nonpositive* degree; there are no positive-degree ghosts, consistent with the absence of gauge symmetry for a bare scheme X. The (-1)-shifted cotangent identification $\mathbf{dCrit}(S) = T^*[-1]X|_{dS}$ endows it with the canonical (-1)-shifted symplectic form of PTVV [9]; unwinding the pairing on polyvector fields gives the odd Poisson bracket, which is the BV antibracket. All ingredients (Koszul resolution, PTVV shifted symplectic structures, the identification of the field/antifield BV complex with functions on the (-1)-shifted cotangent of the critical locus) are established mathematics; see also Costello–Gwilliam's factorization-algebra formulation [8]. Hence the upgrade to status **[S]**. $\square$

Remark 8.8 (Where ghosts come from: schemes give antifields, stacks give ghosts). It is important not to overstate Theorem 8.7. For a bare smooth *scheme* X the Koszul complex $\Lambda^\bullet T_X$ lives entirely in nonpositive cohomological degrees, so the derived critical locus supplies only the *field/antifield* sector (degrees 0 and ≤ -1) of the BV complex — there is no gauge symmetry and hence no ghosts. Ghosts (positive cohomological degree) arise precisely when one resolves a *gauge redundancy*: replacing X by an algebraic stack $[X/G]$ introduces, via the Chevalley–Eilenberg differential of the Lie algebra of G, the positive-degree ghost generators $\mathfrak{g}[1]$ dual to the gauge directions. Thus, in a slogan, *derived geometry supplies the antifields, stacky geometry supplies the ghosts*; the full BV–BRST complex of a gauge theory is the structure sheaf of the derived critical locus *of the stack* $[X/G]$. This is exactly the point at which Theorem 8.3 (stacks retain gauge data) and Theorem 8.7 (derived critical loci) meet.

Remark 8.9 (Why the upgrade matters). The primary source lists "derived critical locus $\leftrightarrow$ BV/BRST field space" as **[S/H]** because, as a bare dictionary slogan, the physical reading was heuristic. Theorem 8.7 shows the field/antifield part of the slogan is in fact a theorem of derived algebraic geometry: the field/antifield BV complex is *literally* the structure sheaf of the derived critical locus, and (by Theorem 8.8) the ghost sector is recovered by the same construction applied to the gauge stack. This is a model of what the whole program aspires to — converting heuristic dictionary entries into theorems where the mathematics permits — and it is the cleanest such upgrade in the algebraic-geometry domain.

9 The Family Realization Pipeline

We assemble the theorems into the promised generalization of Part I's pipeline from a single object to a moduli-indexed family, and we track statuses with Theorem 2.2.

Definition 9.1 (Family realization pipeline)**.** A *family realization pipeline* over a base scheme S is the data

$$\big(\pi\colon \mathcal{X}\to S,\ \underline{\mathrm{Mot}},\ \mathrm{Real}_\alpha,\ \nabla_{\mathrm{GM}},\ \mathrm{Obs}\big)$$

such that $\mathrm{Real}_\alpha \circ \underline{\mathrm{Mot}}$ produces a VHS $\mathcal{H}_\alpha$ on S with Gauss–Manin connection ∇_{GM} and holomorphic section $[\omega_s]$, and the observable $\mathrm{Obs} = \int_{\gamma(\cdot)}$ pairs that section against the flat Betti cycles $\gamma(s)$. The output is a map

$$\mathrm{Obs}^S_\alpha\colon\ \{\text{families over } S\}\ \longrightarrow\ \mathrm{Sol}(\mathcal{L}_{\mathrm{PF}}),\qquad \mathrm{Obs}^S_\alpha(\mathcal{X}) = \Pi(s) \text{ with } \mathcal{L}_{\mathrm{PF}}\Pi = 0,$$

where $\mathrm{Sol}(\mathcal{L}_{\mathrm{PF}})$ is the (finite-dimensional) local solution space of the Picard–Fuchs system of $\mathcal{H}_\alpha$. (The dual statement: integrating a *holomorphic cohomology frame* $e_1,\dots,e_r$ against a *single* flat cycle γ yields a period vector that is a horizontal section of the connection dual to ∇_{GM}; integrating one non-flat form against a flat homology frame yields the non-horizontal, Picard–Fuchs-constrained period functions above.)

Proposition 9.2 (The family pipeline reduces fiberwise to Part I's pipeline; status **[S]**)**.** *For each $s\in S$, evaluating the period function $\mathrm{Obs}^S_\alpha(\mathcal{X}) = \Pi$ at s recovers Part I's $\mathrm{Obs}_\alpha(\mathcal{X}_s) = \mathrm{Obs}(\mathrm{Real}_\alpha(\Phi(\mathcal{X}_s)))$ with $\Phi = \mathrm{Mot}$. Hence the family pipeline is a genuine extension: it agrees with Part I/II fiberwise and adds globally the Picard–Fuchs constraint $\mathcal{L}_{\mathrm{PF}}\Pi = 0$ (equivalently the first-order Gauss–Manin system on the period vector).*

Proof. Immediate from Theorem 8.1(2)–(3): the relative constructions $\underline{\mathrm{Mot}}$, Real_α, Obs are fiberwise the absolute ones, and evaluation $\mathrm{Sol}(\mathcal{L}_{\mathrm{PF}})\to\mathbb{C}$, $\Pi\mapsto\Pi(s)$, gives back the single-object observable at s. □

Proposition 9.3 (Status of the family pipeline)**.** *The family realization pipeline is status* **[S]** *whenever π is smooth projective and the realization channel α is one of {Betti, de Rham, Hodge}; it drops to* **[S/H]** *for the étale channel over a non-closed base field (arithmetic monodromy is only heuristically an "observable"), and to* **[H/P]** *if one further posits a cosmic-Galois action on the period-solution space* $\mathrm{Sol}(\mathcal{L}_{\mathrm{PF}})$ *(a Part II* **[H/P]** *entry).*

Proof. By Theorem 2.2 the status of the composite is the $\wedge$-join of the component statuses. Gauss–Manin/VHS theory over $\mathbb{C}$ is **[S]**; the étale realization's identification with a physical observable is **[S/H]** (arithmetic, not directly measured); the cosmic-Galois covariance is **[H/P]** (Part II, T5). Joining gives the stated results. □

Theorem 9.3 is exactly the epistemic bookkeeping the program demands: it names precisely which realization channels keep the pipeline standard and which push it into heuristic or speculative territory.

10 Worked Examples

10.1 The Legendre family as a family pipeline

Combining Theorem 6.4 with Theorem 9.1: for $\pi\colon\mathcal{E}\to S=\mathbb{P}^1\setminus\{0,1,\infty\}$ the Legendre family, the VHS $\mathcal{H}=R^1\pi_*\underline{\mathbb{Q}}$ has rank 2, the periods $\Pi(\lambda)$ satisfy the hypergeometric Picard–Fuchs operator, and the monodromy representation $\pi_1(S)\to SL_2(\mathbb{Z})$ around $\{0,1,\infty\}$ is the branch-cut data. This is a complete status-**[S]** instance of the entire Part III machinery: a moduli base, a VHS, a Picard–Fuchs equation, and monodromy as physical analytic continuation.

10.2 A quotient stack: $[\mathbb{A}^1/\mathbb{G}_m]$

Let $\mathbb{G}_m$ act on $\mathbb{A}^1$ by scaling. The GIT/coarse quotient is $\operatorname{Spec}\big(\daleth[x]^{\mathbb{G}_m}\big) = \operatorname{Spec}\daleth$, a *single* point (the only invariant functions are the constants, and the only closed orbit is the origin), so the coarse space records nothing beyond a point. The stack $[\mathbb{A}^1/\mathbb{G}_m]$, by contrast, has two points: the open orbit, whose objects have trivial automorphisms, and the origin, whose automorphism group is $\operatorname{Aut}(0) \cong \mathbb{G}_m$ (by Theorem 5.3, $\operatorname{Stab}_{\mathbb{G}_m}(0) = \mathbb{G}_m$). This is the algebraic model of a $U(1)$ gauge theory at a symmetric point: the residual gauge group at the origin is retained by the stack and forgotten by the coarse space, exactly as Theorem 8.3 predicts.

10.3 A derived critical locus: the free scalar

For $X = \mathbb{A}^1$ with coordinate ϕ and action $S(\phi) = \frac{1}{2}m^2\phi^2$, we have $dS = m^2\phi\, d\phi$, $\operatorname{Crit}(S) = \{\phi = 0\}$, and $R/\mathfrak{a} = \daleth[\phi]/(\phi) = \daleth$: the unique on-shell state is $\phi = 0$. The derived critical locus resolves the (here already reduced) intersection by the Koszul complex $\daleth[\phi] \xleftarrow{\cdot m^2\phi} \daleth[\phi]\,\xi$ with ξ the *antifield* in degree -1; the (-1)-shifted symplectic form pairs ϕ with ξ, giving the BV antibracket $\{\phi, \xi\} = 1$. There are no ghosts, since the free scalar has no gauge symmetry (consistent with Theorem 8.8). This is the smallest nontrivial illustration of Theorem 8.7.

11 Formalization: Haskell and Lean

We accompany the paper with executable Haskell encodings of the four load-bearing structures (scheme as functor of points, quotient stack, variation of Hodge structure, positive geometry) and a best-effort Lean 4 sketch of the same type signatures. The Haskell modules compile with GHC and their `main` prints demonstrations of Theorems 5.3 and 8.5(1), and the residue-tree construction. Full sources are in `src/algebraic-geometry-physical-possibility/` and `lean/algebraic-geometry-physical-possibility/`.

11.1 Scheme as functor of points

Following Theorem 3.7 we model a scheme by its functor of points $\mathsf{CRing} \to \mathsf{Set}$, and the on-shell algebra $R/\mathfrak{a}$ by a coordinate ring with a distinguished constraint ideal:

```
-- A commutative ring presented by generators and a constraint ideal.
data CoordRing = CoordRing { generators :: [String], relations :: [Poly] }

-- A scheme exposed via its functor of points R |-> Hom(Spec R, X).
newtype Scheme = Scheme { pointsOver :: CoordRing -> [RPoint] }
```

```
-- The on-shell algebra R/I: functions modulo the ideal of constraints.
-- (idealGens is the record accessor for an Ideal's generating polynomials.)
onShell :: CoordRing -> Ideal -> CoordRing
onShell r i = r { relations = relations r ++ idealGens i }
```

11.2 Quotient stack with stabilizer data (Theorem 5.3)

```
-- A finite group action, enough to compute stabilizers concretely.
data GAction g x = GAction { act :: g -> x -> x, elements :: [g] }

-- Stabilizer of a point: {g | g . x == x}. This is Aut_{[X/G]}(x).
stabilizer :: (Eq x) => GAction g x -> x -> [g]
stabilizer ga x = [ g | g <- elements ga, act ga g x == x ]

-- A quotient stack remembers, for each point, its automorphism group.
data QuotientStack g x = QuotientStack
  { action    :: GAction g x
  , autGroup  :: x -> [g]        -- x |-> Stab_G(x) = Aut_{[X/G]}(x)
  }
```

The demonstration in `Main.hs` builds $[\mathbb{Z}/4 \curvearrowright \mathbb{Z}/4]$ (rotation of a square) and prints the automorphism group at the fixed point, exhibiting Theorem 5.3 concretely: the coarse quotient forgets it, the stack does not.

11.3 Variation of Hodge structure with Gauss–Manin connection

```
-- A VHS over a base: a fiber vector space, a Hodge filtration, and a
-- flat (Gauss-Manin) connection whose FLAT SECTIONS are the Betti cycles
-- (a basis against which the non-flat holomorphic form is measured).
data VHS base = VHS
  { fiber        :: base -> [Double]      -- local system fiber
  , hodgeFiltr   :: base -> [[Double]]    -- F^p, decreasing filtration
  , gaussManin   :: Connection base       -- flat connection nabla
  }

-- Derive the Picard-Fuchs operator by expressing the iterated
-- Gauss-Manin derivatives (nabla^j [omega]) in the finite-rank
-- de Rham basis and reading off the linear dependence. It
-- ANNIHILATES the period function Pi(s) -- the pairing of a flat
-- cycle with the NON-flat form -- which is not a flat section.
derivePicardFuchs :: VHS base -> ODE
derivePicardFuchs = linDep . gmDerivatives
```

11.4 Positive geometry and the recursive residue tree

```
-- A positive geometry: a region, its canonical form, and its boundary
-- strata (themselves positive geometries) -- the recursion of Def 7.1.
data PositiveGeometry = PositiveGeometry
  { canonicalForm :: DiffForm
  , boundaries    :: [PositiveGeometry]
  }

-- The residue tree of Def 7.4 / Theorem 8.5(1): root = canonical form,
-- children = residue trees of the boundary strata.
data Tree a = Node a [Tree a]

residueTree :: PositiveGeometry -> Tree DiffForm
residueTree pg = Node (canonicalForm pg)
                      (map residueTree (boundaries pg))
```

11.5 Lean sketch

The Lean 4 file records idiomatic type signatures for the same structures (scheme via a functor of points, quotient stack with a stabilizer field, VHS with Griffiths transversality, positive geometry with the residue axiom) and states Theorem 8.3 as a signature with a `sorry` placeholder. It is a best-effort sketch, not build-gated:

```
structure QuotientStack (X G : Type) [Group G] [MulAction G X] where
  autGroup : X -> Subgroup G            -- x |-> Stab_G(x) = Aut_{[X/G]}(x)
  coarseMap : X -> Quotient (MulAction.orbitRel G X)

-- T2: a nontrivial stabilizer forces the coarse map to lose information.
theorem stack_finer_than_coarse
    {X G : Type} [Group G] [MulAction G X] (x : X)
    (h : MulAction.stabilizer G x /= (bot : Subgroup G)) :
    True := by trivial   -- best-effort placeholder; see paper Thm 8.3
```

12 Discussion

12.1 How Part III respects the standing limitations

The program carries five standing limitations from Part I; we discharge each for the algebraic-geometry domain.

1. *Not every analogy is a physical theory.* We enforce this with the status labels of Tables 1 and 2 and, sharply, with Theorem 8.5: its part (1) is **[S]** and its part (2) is **[H]**, and we refuse to collapse them.

2. *Motives are not literally "the functions of" an object.* We used Mot only through its *realizations* Real_α (Hodge, Betti, de Rham), which are standard functors; the "functional essence" reading remains a Part II **[H]** slogan and is never invoked as a premise in a proof here.

3. *No single universal functor* Math $\to$ Phys. Our pipeline is domain-local: it is a functor *on the algebraic-geometry domain* $\mathsf{Math}_{\mathsf{AG}}$, fibered over a base S, not a global functor. Theorem 9.1 is explicitly indexed by a family and a channel α.

4. *Not every observable is a period; elliptic/modular/non-Tate structures intervene.* Theorem 6.4 already exhibits an *elliptic* period, and Theorem 8.5(2) is flagged **[H]** precisely because non-Tate geometries obstruct a naive polylogarithmic coaction (Part II, T4). Corollary 8.2 bounds only the *order* of the Picard–Fuchs operator, not the transcendental type of its solutions.

5. *Gauge redundancy $\neq$ physical symmetry.* Theorem 5.5 states this explicitly; Theorem 8.3 makes it geometric (stabilizers are gauge automorphisms, retained by the stack, not new physical states).

12.2 Open problems handed forward

- **Positive geometry = motivic coaction (Theorem 8.5(2)).** The sharpest open problem of this domain: construct, for all amplituhedron-type geometries, a functor from the poset of boundary strata to the comodule of motivic periods intertwining Δ_{geo} with the motivic coproduct Δ. Partial results exist for one-loop integrands and polygon/associahedron geometries; the general case is open ("Hopf algebra of the amplituhedron").

- **Stacky path integrals.** Making Theorem 8.4 quantitative (the precise measure on $\overline{\mathcal{M}}_g$ weighted by $1/|\mathrm{Aut}\,|$) connects Part III to Part IV's TQFT functors and Part VI's stacks-and-descent treatment.

- **Derived enhancement of the whole dictionary.** Theorem 8.7 upgrades one row to **[S]** via derived geometry; systematically deriving the tangent/cotangent-complex rows of Table 2 (BV/BRST for gauge theories) is a natural next step toward Part VI.

12.3 How Part III seeds Part IV

Part IV (algebraic topology: conserved global information) takes the varieties and moduli produced here and studies their *topological invariants* stable under continuous deformation: the (co)homology $H_\bullet(X)$ underlying the Betti realization of Theorem 8.1, the characteristic classes of the bundles in Table 1, and the bordism/TQFT functors that make realization functorial. In particular, the monodromy representation of Theorem 8.2 is a topological (local-system) datum, and the automorphism data of Theorem 8.3 is what Part IV's groupoid-cardinality/TQFT weightings act on. Part III thus supplies the geometric substrate; Part IV supplies its topology.

13 Conclusion

We have formalized the algebraic-geometry layer of the Math→Physics Representation Library as the *geometry of physical possibility*, preserving the program's **[S]**/**[H]**/**[P]** status discipline throughout. The central technical contribution is the extension of Part II's per-object realization pipeline to a *family* pipeline fibered over a moduli base: Gauss–Manin transport realizes periods as the pairing of flat Betti cycles with the non-flat holomorphic form, giving period functions constrained by Picard–Fuchs equations (Theorem 8.1), moduli stacks retain the gauge automorphisms that coarse spaces forget (Theorem 8.3), positive geometries encode factorization through recursive residues that conjecturally match the motivic coaction (Theorem 8.5), and derived critical loci realize the on-shell algebra $R/\mathfrak{a}$ at the BV level, upgrading a heuristic dictionary entry to a theorem (Theorem 8.7).

In the program's compressed slogans:

mathematics is the syntax of physical representation;
motives are functional semantics;
periods are numerical measurements;
coalgebras are decomposition laws.

Part III adds a fifth clause: *geometry is the space in which those measurements vary.* The moduli space is the stage, its singularities are the thresholds, its compactification supplies the asymptotic sectors, and its stackiness is the gauge redundancy: a single algebro-geometric object carrying, in its points, its automorphisms, and its variation of Hodge structure, the entire apparatus of physical possibility. Part IV will endow this stage with its conserved topological invariants.

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