Algebraic Topology: Conserved Global Information

Part IV of *A Math→Physics Representation Library*

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Abstract

This is Part IV of a seven-part *modular* program that formalizes distinct mathematical domains as physical-representation modules within a single representation-stack framework. Building on Part I (the representation stack and realization pipeline $\mathrm{Obs}_\alpha(M) = \mathrm{Obs}(\mathrm{Real}_\alpha(\Phi(M)))$), Part II (motives, periods, amplitudes), and Part III (algebraic geometry, spaces of physical possibility), we develop the thesis that *algebraic topology is the grammar of conserved global information.* We treat (co)homology as the module of observables that are stable under continuous deformation, characteristic classes as quantized topological information carried by gauge bundles, and topological quantum field theory (TQFT) as the first fully rigorous, status-**S** realization of the Part I pipeline in *functorial* form over a category of cobordisms. Our main results are: (T1) that an n-dimensional TQFT $Z\colon \mathrm{Bord}_n \to \mathrm{Vect}_k$ is exactly a representation entry in the sense of Part I, with its symmetric-monoidal structure discharging the Decomposition axiom; (T2) a statement of the cobordism hypothesis as a classification theorem for topological realization channels; (T3) a reading of the Atiyah–Singer index theorem $\mathrm{ind}(D) = \int_X \widehat{A}(TX) \wedge \mathrm{ch}(E)$ as a realization-pipeline identity observable = Real(abstract structure); (T4) that Dijkgraaf–Witten theory instantiates the Locality/descent axiom through group cohomology; and (T5) a bordism-theoretic classification of anomalies and symmetry-protected topological phases. We prove homotopy invariance and functoriality of homology in full, formalize a *conserved global information functor*, and carry the **S**/**H**/**P** epistemic status discipline of the parent program throughout. Companion Haskell code (compiling under GHC) and a best-effort Lean sketch encode chain complexes, homology, characteristic classes, and the TQFT axioms. Part IV is where category theory first appears as the ambient grammar, and it seeds Part V.

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1 Introduction

1.1 The modular program and where Part IV sits

The parent program, *A Math→Physics Representation Library*, advances a single organizing claim: many physical quantities are not merely *described by* mathematics but are *obtained as realizations* of structured mathematical information. The organizing device is the *realization pipeline*

$$\begin{gathered} M \xrightarrow{\Phi} \Phi(M) \xrightarrow{\mathrm{Real}_\alpha} \text{physical representation} \xrightarrow{\mathrm{Obs}} \text{measurement data}, \\ \mathrm{Obs}_\alpha(M) = \mathrm{Obs}\big(\mathrm{Real}_\alpha(\Phi(M))\big), \end{gathered} \tag{1}$$

together with a three-level epistemic status system that every module must carry:

Label	Meaning	Canonical example
S	standard use in mathematics or physics	TQFT as a functor $\mathrm{Bord}_n \to \mathrm{Vect}$
H	strong heuristic dictionary entry	motive as "functional essence"
P	speculative philosophical ontology	reality as motivic-categorical realization

Composite labels (**S**/**H**, **H**/**P**) appear when a construction is standard as pure mathematics but heuristic or speculative in its *physical* reading. We use the labels verbatim; they are part of the formalism, not decoration.

The program is explicitly *modular* rather than unified: each part is readable and defensible on its own, and composition is exhibited as a hierarchical building process. The ladder proceeds

$$\underbrace{\mathrm{I}}_{\text{stack}} \to \underbrace{\mathrm{II}}_{\text{periods}} \to \underbrace{\mathrm{III}}_{\text{geometry}} \to \underbrace{\mathrm{IV}^\star}_{\text{topology}} \to \underbrace{\mathrm{V}}_{\text{cat./HoTT}} \to \underbrace{\mathrm{VI}}_{\text{stacks/gauge/QHA}} \to \text{Synth.},$$

where $\mathrm{IV}^\star$ denotes the present paper. Part III equips the "spaces of physical possibility" (varieties, moduli) with a geometry. Part IV equips those same spaces with *topological invariants* stable under continuous deformation, and it is the first place where a *genuinely functorial, status-***S** instance of the pipeline (1) appears: a topological quantum field theory is literally a functor. This is the historical and conceptual bridge to Part V, where functoriality is abstracted into the universal grammar of categories and homotopy type theory.

1.2 Thesis of Part IV

We defend, formalize, and where possible prove the slogan

algebraic topology is the grammar of conserved global information.

Concretely: a homology class is a conserved extended structure; a cohomology class is a stable observable insensitive to local deformation; characteristic classes encode quantized topological information carried by bundles; bordism groups and generalized cohomology classify topological phases and anomalies; and TQFT packages all of this as a symmetric monoidal functor from cobordisms to an algebraic target. The recurring physical motif is *conservation*: what survives continuous deformation is exactly what algebraic topology measures, and this is the topological face of the parent program's Decomposition and Locality axioms.

1.3 Contributions

1. **A representation-stack account of (co)homology** (Section 3). We define a *conserved global information functor* $H_\bullet\colon \mathrm{hTop} \to \mathrm{GrAb}$ and prove homotopy invariance and functoriality in full, giving a precise sense in which topological observables are *conserved* under physical deformation (Theorems 3.8 and 3.10).

2. **TQFT as a status-S realization entry** (Section 5). We show an n-dimensional TQFT is exactly a Part I representation entry, and that symmetric monoidality is the Decomposition axiom over cobordisms (Theorem 5.3); we state the cobordism hypothesis as a classification theorem for topological realization channels (Theorem 5.5).

3. **The index theorem as a pipeline identity** (Section 4). We reframe Atiyah–Singer as an instance of observable = Real(abstract structure) (Theorem 4.6), the strongest available status-**S** anchor.

4. **Discrete gauge theory and locality** (Section 6). We show Dijkgraaf–Witten theory instantiates the Locality/descent axiom via a group cocycle, with triangulation independence coming from Pachner-move invariance (Theorem 6.2).

5. **Bordism classification of anomalies and phases** (Section 7). We record the Freed–Hopkins classification of invertible field theories and its physical reading as a status-**S** / **S**/**H** statement (Theorem 7.3).

6. **Formalization** (Section 9). Companion Haskell code (compiling under GHC) and a Lean sketch encode chain complexes, homology, characteristic classes and TQFT axioms, extending the parent program's formal-verification bridge.

1.4 Epistemic status of Part IV

Part IV is unusually rich in status-**S** content: TQFT, the index theorem, Dijkgraaf–Witten theory, and the bordism classification of invertible field theories are all standard mathematics or mathematical physics. The heuristic (**H**) content lies in the *physical readings* of dictionary entries (e.g. "coboundary = pure gauge"), and the speculative (**P**) content is confined to the ontological framing inherited from the parent program. We flag labels at every dictionary entry (Section A) and at each theorem.

1.5 Outline

Section 2 recalls the representation stack and pipeline from Part I in the precise form used here. Section 3 develops (co)homology as conserved global information. Section 4 treats characteristic classes and the index theorem. Section 5 develops TQFT and the cobordism hypothesis. Section 6 treats Dijkgraaf–Witten theory; Section 7 treats bordism and SPT phases. Section 8 collects the main results. Section 9 describes the companion code. Section 10 discusses limitations and connections to Parts III and V, and Section 11 concludes. Section A reproduces the algebraic-topology dictionary with status labels.

2 Mathematical Framework: the representation stack, recalled

We recall from Part I only what Part IV uses, in a self-contained form.

Definition 2.1 (Domains and targets)**.** Let Dom be a (small) category of *mathematical domains*. For each $d \in \mathsf{Dom}$ let Math_d be a category (or higher category) of mathematical structures in that domain. Let Phys be a category of *physical representation targets*: state spaces, observable algebras, process categories, gauge moduli, quantum code spaces, and measurement outcomes. In Part IV the relevant domains are Top (topological spaces), Man_n (closed manifolds and cobordisms), and the associated Bord_n.

Definition 2.2 (Representation entry)**.** A *representation entry* is a quadruple $E = (M, P, \tau, \sigma)$ where $M \in \mathsf{Math}_d$ is a mathematical structure, $P \in \mathsf{Phys}$ a physical representation, τ a semantic translation datum (a functor, a natural transformation, or a weaker relation), and σ is the epistemic status label (**S**, **H**, or **P**).

Definition 2.3 (Realization pipeline)**.** A *realization pipeline* on categories $M \xrightarrow{\Phi} I \xrightarrow{\mathrm{Real}_\alpha} R \xrightarrow{\mathrm{Obs}} O$ is a composable triple of functors; the associated *observable functor* is $\mathrm{Obs}_\alpha := \mathrm{Obs} \circ \mathrm{Real}_\alpha \circ \Phi$, and we write $\mathrm{Obs}_\alpha(M) = \mathrm{Obs}(\mathrm{Real}_\alpha(\Phi(M)))$ as in (1). The index α ranges over *realization channels*; in Part IV the channels of interest are *topological*: singular/de Rham (co)homology, characteristic classes, TQFT evaluation, and the analytic index.

Part I isolates four axioms. We restate the two that Part IV discharges concretely.

Definition 2.4 (Locality and descent axiom)**.** Physical representations must be locally assignable and globally coherent; a failure of descent is an obstruction, an anomaly, or a missing boundary datum.

Definition 2.5 (Decomposition axiom)**.** Compositional or period-like observables admit a decomposition operation (boundary, coproduct, coaction, spectral sequence, factorization, or gluing) revealing lower-level informational pieces.

The master diagram of Part I, in the form we use, is

$$\begin{tikzcd} M' \arrow[d, dashed, "\sim"', "\text{gauge / equiv. / duality}"] & & & \\ M \arrow[r, "\Phi"] & \Phi(M) \arrow[r, "\mathrm{Real}_\alpha"] & \text{physical rep.} \arrow[r, "\mathrm{Obs}"] & \text{observable} \end{tikzcd}$$

The dashed identification $M \sim M'$ encodes the Equivalence axiom: mathematical descriptions related by a gauge or equivalence relation determine the same physical content at the chosen observational level. Part IV realizes the dashed arrow twice: as homotopy of spaces (Section 3) and as coboundary shift of a Dijkgraaf–Witten cocycle (Section 6).

Remark 2.6 (Status of the framework). Definitions 2.1 to 2.3 are status-**S** as pure category theory; their *physical* interpretation carries the label of the individual entry. Part IV's contribution is to populate the framework with entries whose τ is an *actual functor*, i.e. status-**S** entries, in contrast to the heuristic entries that dominate Parts II–III.

3 (Co)homology as conserved global information

We now develop the core of Part IV: algebraic topology as the module of observables conserved under continuous deformation. We proceed from chain complexes to homology, prove functoriality and homotopy invariance, dualize to cohomology, and read the results physically.

3.1 Chain complexes and the boundary operator

Definition 3.1 (Chain complex)**.** A *chain complex* of abelian groups is a sequence

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

of abelian groups C_n ($n \geq 0$) and homomorphisms $\partial_n \colon C_n \to C_{n-1}$ satisfying the *fundamental relation*

$$\partial_n \circ \partial_{n+1} = 0 \qquad \text{for all } n. \tag{2}$$

Elements of $Z_n := \ker \partial_n$ are *n-cycles*; elements of $B_n := \operatorname{im} \partial_{n+1}$ are *n-boundaries*. A morphism of chain complexes (a *chain map*) $f_\bullet \colon C_\bullet \to D_\bullet$ is a family $f_n \colon C_n \to D_n$ with $\partial_n^D f_n = f_{n-1} \partial_n^C$.

Lemma 3.2 (Boundaries are cycles)**.** *For every chain complex, $B_n \subseteq Z_n$, so $H_n := Z_n / B_n$ is a well-defined abelian group.*

Proof. Let $b \in B_n$, say $b = \partial_{n+1} c$ for some $c \in C_{n+1}$. Then by (2), $\partial_n b = \partial_n \partial_{n+1} c = 0$, so $b \in \ker \partial_n = Z_n$. Hence $B_n \subseteq Z_n$. As C_n is abelian, B_n is a (normal) subgroup of Z_n and the quotient Z_n / B_n is an abelian group. □

Definition 3.3 (Homology)**.** The *n-th homology group* of $C_\bullet$ is $H_n(C_\bullet) := Z_n / B_n$. The class of a cycle $z \in Z_n$ is written $[z]$. A chain complex is *exact* at C_n if $H_n(C_\bullet) = 0$, i.e. $Z_n = B_n$.

For a topological space X we take $C_\bullet = C_\bullet(X)$ the singular chain complex, with $C_n(X)$ the free abelian group on continuous maps $\sigma \colon \Delta^n \to X$ and $\partial_n \sigma = \sum_{i=0}^n (-1)^i \, \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$ the alternating face sum. Then $H_n(X) := H_n(C_\bullet(X))$ is singular homology.

Lemma 3.4 (Singular boundary squares to zero)**.** *The singular boundary satisfies $\partial_n \partial_{n+1} = 0$, so $C_\bullet(X)$ is a chain complex in the sense of Definition 3.1.*

Proof. It suffices to check on a single simplex $\sigma\colon \Delta^{n+1} \to X$. Writing $\sigma_{[\hat{i}]}$ for the i-th face,

$$\partial_n \partial_{n+1} \sigma = \sum_{i=0}^{n+1} (-1)^i \, \partial_n \sigma_{[\hat{i}]} = \sum_{i=0}^{n+1} \sum_{j=0}^{n} (-1)^i (-1)^j \, (\sigma_{[\hat{i}]})_{[\hat{j}]}.$$

The face maps satisfy the simplicial identity $(\sigma_{[\hat{i}]})_{[\hat{j}]} = (\sigma_{[\hat{j}]})_{[\widehat{i-1}]}$ for $j < i$. Splitting the double sum into $j < i$ and $j \geq i$ and applying this identity shows the two pieces cancel term by term after reindexing, giving 0. □

3.2 Functoriality: the conserved-information functor

Proposition 3.5 (Homology is functorial). *A continuous map $f\colon X \to Y$ induces a chain map $f_\#\colon C_\bullet(X) \to C_\bullet(Y)$ by $f_\#(\sigma) = f \circ \sigma$, hence homomorphisms $f_*\colon H_n(X) \to H_n(Y)$ for all n. This assignment satisfies $(\mathrm{id}_X)_* = \mathrm{id}$ and $(g \circ f)_* = g_* \circ f_*$; thus $H_n(-)$ is a functor $\mathrm{Top} \to \mathrm{Ab}$.*

Proof. For a singular simplex $\sigma\colon \Delta^n \to X$ and $0 \leq i \leq n$ we have $(f \circ \sigma)_{[\hat{i}]} = f \circ \sigma_{[\hat{i}]}$, so $f_\#$ commutes with the face maps and hence with ∂; thus $f_\#$ is a chain map. A chain map sends cycles to cycles and boundaries to boundaries, so it descends to $f_*\colon H_n(X) \to H_n(Y)$. Functoriality of $(-)_\#$ is immediate from $(g \circ f) \circ \sigma = g \circ (f \circ \sigma)$ and passes to homology. □

The physical content of Part IV depends on more than functoriality. Because H_n is *homotopy invariant*, it cannot distinguish continuously deformable maps, and this is what we mean by conservation under deformation.

Definition 3.6 (Chain homotopy). Two chain maps $f_\bullet, g_\bullet\colon C_\bullet \to D_\bullet$ are *chain homotopic* if there is a family $P_n\colon C_n \to D_{n+1}$ with

$$\partial^D_{n+1} P_n + P_{n-1} \partial^C_n = g_n - f_n \qquad \text{for all } n. \tag{3}$$

Lemma 3.7 (Chain homotopy ⇒ equality on homology). *If $f_\bullet$ and $g_\bullet$ are chain homotopic, then $f_* = g_*$ on homology.*

Proof. Let $z \in Z_n$, so $\partial z = 0$. Applying (3),

$$(g_n - f_n)(z) = \partial_{n+1} P_n(z) + P_{n-1} \partial_n(z) = \partial_{n+1}\big(P_n(z)\big) + P_{n-1}(0) = \partial_{n+1}\big(P_n(z)\big) \in B_n.$$

Hence $g_n(z)$ and $f_n(z)$ differ by a boundary, so $[g_n(z)] = [f_n(z)]$ in $H_n(D_\bullet)$; that is, $g_* = f_*$. □

Theorem 3.8 (Homotopy invariance of homology). *If $f, g\colon X \to Y$ are homotopic continuous maps, then $f_* = g_*\colon H_n(X) \to H_n(Y)$ for all n. Consequently a homotopy equivalence induces isomorphisms on all homology groups, and H_n descends to a functor*

$$H_n\colon \mathrm{hTop} \longrightarrow \mathrm{Ab}$$

on the homotopy category hTop *(spaces and homotopy classes of maps).*

Proof. A homotopy $F\colon X \times [0,1] \to Y$ from f to g induces a chain homotopy between $f_\#$ and $g_\#$ via the *prism operator* $P_n\colon C_n(X) \to C_{n+1}(Y)$ obtained by subdividing the prism $\Delta^n \times [0,1]$ into $(n+1)$-simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$ (with $v_j = (x,0)$, $w_j = (x,1)$ vertices) and setting $P_n(\sigma) = \sum_{i=0}^n (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,w_n]}$. A direct computation of $\partial P + P\partial$ using the simplicial identities yields $g_\# - f_\#$, i.e. (3). By Lemma 3.7, $f_* = g_*$. If f is a homotopy equivalence with inverse h, then $f_*h_* = (fh)_* = (\mathrm{id})_* = \mathrm{id}$ and similarly $h_*f_* = \mathrm{id}$, so f_* is an isomorphism. Since H_n agrees on homotopic maps, it factors through hTop. □

We package this as the central object of Part IV.

Definition 3.9 (Conserved global information functor)**.** The *conserved global information functor* is

$$H_\bullet\colon \mathrm{hTop} \longrightarrow \mathrm{GrAb}, \qquad X \longmapsto \big(H_n(X)\big)_{n\geq 0},$$

valued in graded abelian groups, assembled from Theorem 3.8. A *topological observable* of X is an element of $H^n(X)$ (see Definition 3.11); its value on a cycle is a *conserved charge.*

Theorem 3.10 (Conservation principle)**.** *Let $\{X_t\}_{t\in[0,1]}$ be a continuous family of spaces realized as the fibers of a fibration $E \to [0,1]$ over the contractible base $[0,1]$. Then for all n the groups $H_n(X_t)$ are canonically isomorphic, and any topological observable $\omega \in H^n(X_0)$ has a unique continuation $\omega_t \in H^n(X_t)$ with $\langle \omega_t, [z_t]\rangle$ independent of t for a continuously varying cycle class $[z_t]$. In this precise sense topological observables are conserved under continuous physical deformation.*

Proof. A fibration over a contractible base is fiber-homotopy equivalent to the trivial fibration $X_0 \times [0,1] \to [0,1]$; in particular any two fibers X_s, X_t are homotopy equivalent, and parallel transport along $[0,1]$ furnishes a canonical homotopy class of equivalence $X_0 \simeq X_t$. By Theorem 3.8 the induced maps $H_n(X_0) \xrightarrow{\cong} H_n(X_t)$ are isomorphisms; dualizing (Proposition 3.12) gives the corresponding isomorphisms on H^n and hence the continuation ω_t. The pairing $\langle \omega_t, [z_t]\rangle$ is preserved because both ω_t and $[z_t]$ are images of $\omega_0, [z_0]$ under the same transport isomorphism, which is compatible with the (co)homology pairing (Definition 3.13). Thus the pairing is constant in t. □

3.3 Cohomology, closed and exact forms, and gauge

Definition 3.11 (Singular cohomology)**.** For an abelian group A, the *singular cochain complex* of X is $C^n(X;A) := \mathrm{Hom}(C_n(X), A)$ with coboundary $\delta^n = \mathrm{Hom}(\partial_{n+1}, A)$, i.e. $(\delta\varphi)(c) = \varphi(\partial c)$. Then $\delta^{n+1}\delta^n = 0$ (dual to (2)), and the *n-th cohomology group* is $H^n(X;A) := \ker \delta^n / \operatorname{im} \delta^{n-1}$. Cocycles are Z^n, coboundaries are B^n.

Proposition 3.12 (Cohomology is a contravariant functor)**.** $H^n(-;A)\colon \mathrm{hTop}^{\mathrm{op}} \to \mathrm{Ab}$ *is a functor: a map $f\colon X \to Y$ induces $f^* = \mathrm{Hom}(f_\#, A)\colon H^n(Y;A) \to H^n(X;A)$, with $(\mathrm{id})^* = \mathrm{id}$, $(gf)^* = f^*g^*$, and f^* depending only on the homotopy class of f.*

Proof. $\mathrm{Hom}(-, A)$ is a contravariant additive functor, so it sends the chain map $f_\#$ to a cochain map f^* and reverses composition; homotopy invariance is dual to Theorem 3.8 because $\mathrm{Hom}(-, A)$ sends a chain homotopy (3) to a cochain homotopy. □

Definition 3.13 (Evaluation pairing)**.** The *Kronecker pairing* $\langle -, - \rangle \colon H^n(X; A) \times H_n(X) \to A$ is $\langle [\varphi], [z] \rangle := \varphi(z)$. It is well defined: if $\varphi = \delta\psi$ then $\varphi(z) = \psi(\partial z) = 0$ for a cycle z, and if $z = \partial w$ then $\varphi(z) = (\delta\varphi)(w) = 0$ for a cocycle φ.

The de Rham incarnation makes the "conserved field" reading transparent.

Definition 3.14 (de Rham cohomology)**.** For a smooth manifold X, let $\Omega^n(X)$ be the smooth n-forms with exterior derivative $d \colon \Omega^n \to \Omega^{n+1}$, $d^2 = 0$. A form is *closed* if $d\omega = 0$ and *exact* if $\omega = d\alpha$. The *de Rham cohomology* is $H^n_{\mathrm{dR}}(X) := \ker(d|_{\Omega^n}) / \operatorname{im}(d|_{\Omega^{n-1}})$.

Proposition 3.15 (Conservation and gauge, de Rham form)**.** *Let $\omega \in \Omega^n(X)$ be a field strength. Then:*

(i) $d\omega = 0$ *(a conservation law / Bianchi identity) is the statement that ω is a cocycle; its class $[\omega] \in H^n_{\mathrm{dR}}(X)$ is a deformation-invariant observable.*

(ii) a shift $\omega \mapsto \omega + d\alpha$ leaves the class $[\omega]$ unchanged, so $[\omega]$ is the gauge-invariant flux and an exact part $d\alpha$ carries the topologically trivial class $[\, d\alpha \,] = 0$ (status **H***). We caution that this is* global *cohomological triviality: at the level of a potential A with $\omega = dA$, the genuine local gauge redundancy is $A \mapsto A + d\lambda$, which leaves ω pointwise invariant, whereas a nonzero exact field strength $d\alpha$ need not vanish pointwise. Thus "pure gauge" here means null in cohomology, not $\omega = 0$.*

(iii) For a closed oriented n-cycle Γ, the flux $\Phi(\Gamma) := \int_\Gamma \omega$ depends only on $[\omega]$ and the homology class $[\Gamma]$, by Stokes' theorem; it is the conserved charge measured over Γ.

Proof. (i)–(ii) are immediate from Definition 3.14: $d\omega = 0$ means ω is a de Rham cocycle, and $d(\omega + d\alpha) = d\omega = 0$ with $[\omega + d\alpha] = [\omega]$ since $d\alpha$ is exact. (iii): if $\omega' = \omega + d\alpha$ and $\Gamma' = \Gamma + \partial S$, then by Stokes $\int_{\Gamma'} \omega' = \int_\Gamma \omega + \int_\Gamma d\alpha + \int_{\partial S} \omega + \int_{\partial S} d\alpha$, and each correction integral vanishes because Γ is closed ($\int_\Gamma d\alpha = \int_{\partial\Gamma} \alpha = 0$), ω is closed ($\int_{\partial S} \omega = \int_S d\omega = 0$), and likewise for the last term. Thus Φ descends to $H^n_{\mathrm{dR}} \times H_n$. □

By the de Rham theorem, $H^n_{\mathrm{dR}}(X) \cong H^n(X; \mathbb{R})$ with the pairing of Definition 3.13 matching integration; so the de Rham and singular pictures are the same status-**S** observable module in two realization channels.

3.4 Exact sequences: conservation and obstruction bookkeeping

Proposition 3.16 (Long exact sequence of a pair)**.** *For a pair (X, B) there is a natural long exact sequence*

$$\cdots \longrightarrow H_n(B) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, B) \xrightarrow{\partial} H_{n-1}(B) \longrightarrow \cdots$$

in which $\operatorname{im} i_* = \ker j_*$, $\operatorname{im} j_* = \ker \partial$, and $\operatorname{im} \partial = \ker i_*$. Physically, ∂ is the boundary/defect charge-transfer map: a bulk class with nontrivial ∂ deposits conserved charge on the boundary B.

Proof. This is the homology long exact sequence associated to the short exact sequence of chain complexes $0 \to C_\bullet(B) \to C_\bullet(X) \to C_\bullet(X,B) \to 0$, where $C_\bullet(X,B) := C_\bullet(X)/C_\bullet(B)$. Exactness is the standard homological snake lemma / zig-zag construction: the connecting map ∂ sends the class of a relative cycle c (so $\partial c \in C_{n-1}(B)$) to $[\partial c] \in H_{n-1}(B)$, and the three exactness statements are verified by diagram chase. Naturality follows because a map of pairs induces a map of the short exact sequences. ☐

Proposition 3.17 (Mayer–Vietoris: reconstructing global physics from patches)**.** *If $X = U \cup V$ with U, V open, there is a long exact sequence*

$$\cdots \longrightarrow H_n(U \cap V) \longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(X) \xrightarrow{\partial} H_{n-1}(U \cap V) \longrightarrow \cdots$$

This is the topological form of the Locality/descent axiom (Definition 2.4): global conserved information on X is reconstructed from local information on U, V together with the gluing data on the overlap $U \cap V$; the connecting map ∂ is the obstruction to naive gluing.

Proof. Apply the homology long exact sequence to the short exact sequence $0 \to C_\bullet(U \cap V) \xrightarrow{(i,-j)} C_\bullet(U) \oplus C_\bullet(V) \xrightarrow{k+l} C_\bullet(U+V) \to 0$, and use that the inclusion $C_\bullet(U+V) \hookrightarrow C_\bullet(X)$ of sums of chains supported in U or V induces isomorphisms on homology (small-simplices / barycentric-subdivision theorem). Exactness is again the zig-zag lemma. ☐

Remark 3.18 (Descent reading). Proposition 3.17 is exactly a descent statement: the sequence exhibits $H_\bullet(X)$ as glued from $H_\bullet(U), H_\bullet(V)$ along $H_\bullet(U \cap V)$, up to the correction measured by ∂. When $\partial = 0$ the local data glue on the nose (a *sheaf*-like situation); when $\partial \neq 0$ they glue only up to the obstruction (a *stack*-like situation). This is the first appearance in the ladder of the sheaf/stack dichotomy that Part VI makes precise.

3.5 Poincaré duality and cup product

Theorem 3.19 (Poincaré duality)**.** *Let X be a closed oriented n-manifold with fundamental class $[X] \in H_n(X)$. Then cap product with $[X]$ is an isomorphism*

$$- \frown [X] \colon H^k(X) \xrightarrow{\cong} H_{n-k}(X) \qquad (0 \le k \le n).$$

Physically: a codimension-k defect (a class in H_{n-k}) is equivalent data to a degree-k field-strength observable (a class in H^k); defects and dual field strengths carry the same conserved information.

Proof. This is the classical Poincaré duality theorem; we recall the mechanism. Cap product $\frown\colon H^k(X) \otimes H_n(X) \to H_{n-k}(X)$ is defined on the chain level by $\varphi \frown \sigma = \varphi(\sigma|_{\text{front } k})\, \sigma|_{\text{back } n-k}$ and descends to (co)homology. For a closed oriented manifold, capping with the fundamental class is shown to be an isomorphism by the standard local-to-global argument, which is naturally phrased using *compactly supported* cohomology H^k_c: unlike ordinary cohomology (which has $H^n(\mathbb{R}^n) = 0$), one has $H^k_c(\mathbb{R}^n) \cong \mathbb{Z}$ concentrated in degree $k = n$, and the duality $H^k_c(\mathbb{R}^n) \cong H_{n-k}(\mathbb{R}^n)$ holds for $\mathbb{R}^n$ and half-spaces (where both sides are computable). One then propagates along a good cover by an induction using the Mayer–Vietoris sequences (Proposition 3.17) on both sides and the five lemma; the naturality of $\frown$

makes the induction squares commute. For the closed manifold X one has $H^k_c(X) = H^k(X)$, yielding the stated form. $\square$

Definition 3.20 (Cup product)**.** The *cup product* $\smile\colon H^k(X;R)\otimes H^\ell(X;R) \to H^{k+\ell}(X;R)$ is induced by $(\varphi \smile \psi)(\sigma) = \varphi(\sigma\,|_{\text{front}}) \cdot \psi(\sigma\,|_{\text{back}})$, making $H^\bullet(X;R)$ a graded-commutative ring. Physically the cup product is the *interaction* of two charge/flux observables, and its graded commutativity $\alpha \smile \beta = (-1)^{|\alpha||\beta|}\beta \smile \alpha$ (**S**/**H**) encodes the statistics of the corresponding excitations.

Example 3.21 (Conserved charge on the torus)**.** For $X = T^2 = S^1\times S^1$ one has $H^1(T^2;\mathbb{Z}) \cong \mathbb{Z}^2$ generated by classes a, b dual to the two circles, with $a \smile a = b \smile b = 0$ and $a \smile b = -b \smile a$ generating $H^2(T^2;\mathbb{Z}) \cong \mathbb{Z}$. Two *distinct* conserved observables live here and must not be confused.

(i) A *flat* gauge field ($F = 0$) is classified up to gauge by its two holonomies, i.e. by a class in $H^1(T^2;\mathrm{U}(1)) \cong \mathrm{U}(1) \times \mathrm{U}(1)$ (Wilson loops around the two cycles). Such a connection has vanishing curvature and hence *trivial* first Chern number; the moduli space of flat connections detects only the H^1-data.

(ii) A generally *non-flat* complex line bundle $L \to T^2$ carries a quantized magnetic flux, the first Chern number $\int_{T^2} \frac{F}{2\pi} = \langle c_1(L), [T^2]\rangle \in \mathbb{Z}$ with $c_1(L) \in H^2(T^2;\mathbb{Z}) \cong \mathbb{Z}$; a nonzero value *forces* $F \neq 0$, so this observable is disjoint from case (i).

It is precisely the nondegenerate cup pairing $\langle a \smile b, [T^2]\rangle = 1$ that makes $H^2(T^2;\mathbb{Z}) \cong \mathbb{Z}$ one-dimensional and thereby able to host the integer flux of case (ii). Both are toy models underlying the quantum-Hall and surface-code phenomenology revisited in Part VI.

4 Characteristic classes and the index theorem

Characteristic classes are the module of *quantized* topological information: they assign to a gauge bundle a cohomology class of its base, natural under pullback, and their integrals are integer- or rational-valued conserved charges.

4.1 Classifying spaces and characteristic classes

Definition 4.1 (Principal bundle and classifying space)**.** A *principal G-bundle* over X is a fiber bundle $\pi\colon P \to X$ with a free, fiber-transitive right G-action. Isomorphism classes of principal G-bundles over a paracompact X are in natural bijection with homotopy classes of maps $[X, BG]$, where BG is the *classifying space* (the base of the universal bundle $EG \to BG$); the bijection sends $f\colon X \to BG$ to the pullback f^*EG.

Definition 4.2 (Characteristic class)**.** A *characteristic class* for G-bundles with coefficients in R is a natural transformation $c\colon [-, BG] \Rightarrow H^*(-;R)$; equivalently, by Definition 4.1 and the Yoneda lemma, an element $c \in H^*(BG;R)$, with the class of a bundle f^*EG given by $c(f^*EG) = f^*c$.

Proposition 4.3 (Naturality is conservation). *For any continuous $g\colon Y \to X$ and any principal G-bundle $P \to X$, $c(g^*P) = g^*c(P)$. In particular characteristic numbers $\int_X c(P)$ of closed manifolds are homotopy/cobordism invariants: they are conserved under continuous deformation of the bundle and of the base.*

Proof. Naturality is Definition 4.2 directly: if $P = f^*EG$ then $g^*P = (f \circ g)^*EG$ has class $(f \circ g)^*c = g^*f^*c = g^*c(P)$. The characteristic number statement follows because $\int_X\colon H^n(X) \to R$ (for X closed oriented n-manifold) is a homotopy invariant of the pair (X, class) by Theorem 3.8 and Proposition 3.12. □

Example 4.4 (Chern classes and quantized flux). For $G = \mathrm{U}(k)$ one has the polynomial ring

$$H^*(B\mathrm{U}(k); \mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_k], \qquad c_i \in H^{2i},$$

whose generators c_i are the *Chern classes*. For a complex line bundle $L \to X$, $c_1(L) \in H^2(X; \mathbb{Z})$ is the quantized first Chern number; over a closed surface Σ, $\int_\Sigma c_1(L) = \frac{1}{2\pi}\int_\Sigma F \in \mathbb{Z}$ is the Dirac-quantized magnetic monopole charge / total flux. The *Chern character*

$$\mathrm{ch}(E) = \mathrm{rk}(E) + c_1 + \tfrac{1}{2}(c_1^2 - 2c_2) + \cdots \in H^{\mathrm{even}}(X; \mathbb{Q})$$

is a ring homomorphism $\mathrm{ch}\colon K^0(X) \to H^{\mathrm{even}}(X; \mathbb{Q})$ turning K-theory charges into cohomological ones. All entries here are status-**S**.

Example 4.5 (Stiefel–Whitney classes and spin). For $G = \mathrm{O}(k)$, the *Stiefel–Whitney classes* $w_i \in H^i(X; \mathbb{Z}/2)$ obstruct structure: $w_1(TX) = 0$ iff X is orientable, and (given $w_1 = 0$) $w_2(TX) = 0$ iff X admits a *spin structure*, the condition under which fermions can be globally defined. Thus w_2 is a conserved $\mathbb{Z}/2$ obstruction whose vanishing is a consistency condition for fermionic matter (**S**).

4.2 The index theorem as a realization-pipeline identity

Theorem 4.6 (Atiyah–Singer as observable = Real(abstract structure)). *Let X be a closed even-dimensional spin manifold with spinor bundle $S = S^+ \oplus S^-$ (the chiral splitting), and $E \to X$ a complex vector bundle with connection; let $D_E^+\colon \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$ be the associated chiral twisted Dirac operator. Then the analytic index equals a purely topological expression:*

$$\mathrm{ind}(D_E^+) \;=\; \dim\ker D_E^+ - \dim\ker D_E^- \;=\; \int_X \widehat{A}(TX) \wedge \mathrm{ch}(E) \;\in\; \mathbb{Z}, \tag{4}$$

where D_E^- is the formal adjoint (so $\ker D_E^- = \operatorname{coker} D_E^+$). Reading the left side as an observable Obs *(a net chiral zero-mode count / anomaly coefficient) and the right side as* $\mathrm{Real}_{\mathrm{top}}\big(\Phi(X,E)\big)$ *with $\Phi(X,E)$ the topological data $\widehat{A}(TX), \mathrm{ch}(E)$, (4) is an instance of the Part I master formula* $\mathrm{Obs}_{\mathrm{top}}(M) = \mathrm{Obs}(\mathrm{Real}_{\mathrm{top}}(\Phi(M)))$ *with realization channel $\alpha = \mathrm{top}$. Status* **S**.

Proof. Equation (4) is the Atiyah–Singer index theorem [10], which we do not reprove; we verify only the pipeline reading. Set $\Phi(X,E) := (\widehat{A}(TX), \mathrm{ch}(E)) \in H^\bullet(X; \mathbb{Q})$, a functor of the topological data (natural under bundle pullback by Proposition 4.3). Let $\mathrm{Real}_{\mathrm{top}}$ be the integration channel $\mathrm{Real}_{\mathrm{top}}(\eta) = \int_X \eta$ applied to the top-degree component of the product $\widehat{A}(TX) \wedge \mathrm{ch}(E)$, valued in $\mathbb{Q}$; and let Obs be the identity on $\mathbb{Z} \subseteq \mathbb{Q}$. Then $\mathrm{Obs}(\mathrm{Real}_{\mathrm{top}}(\Phi(X,E))) = \int_X \widehat{A}(TX) \wedge \mathrm{ch}(E)$, which by (4) equals $\mathrm{ind}(D_E^+)$, the analytic observable. Each of $\Phi, \mathrm{Real}_{\mathrm{top}}, \mathrm{Obs}$ is functorial (naturality, linearity of the integral, inclusion), so $\mathrm{Obs}_{\mathrm{top}} = \mathrm{Obs} \circ \mathrm{Real}_{\mathrm{top}} \circ \Phi$ is a realization pipeline in the sense of Definition 2.3, and (4) is the assertion that it computes the index. Integrality of the right side is a nontrivial consequence, exactly the content of the theorem. □

Remark 4.7 (Why the index theorem is the ideal status-**S** anchor). Theorem 4.6 imports no new mathematics: its physical reading ("topology → analysis: zero modes and anomalies from topology") *is* the theorem. This makes it the least contestable entry in the entire library and the model we hold other, more heuristic entries against.

5 Topological quantum field theory: functorial realization

A TQFT sits at the conceptual center of Part IV: it is a realization pipeline that is *itself a functor* over a category of cobordisms. It is the first status-**S** instance of the Part I pipeline in which Real_α is a genuine functor rather than a heuristic translation rule, and for that reason Part IV is where category theory enters the ladder as ambient grammar.

5.1 The bordism category

Definition 5.1 (Bordism category)**.** Fix $n \geq 1$. The *bordism category* Bord_n has:

- **objects**: closed oriented $(n-1)$-manifolds Σ;
- **morphisms** $\Sigma_0 \to \Sigma_1$: oriented cobordisms, i.e. (diffeomorphism classes rel boundary of) compact oriented n-manifolds W with $\partial W \cong \overline{\Sigma_0} \sqcup \Sigma_1$;
- **composition**: gluing of cobordisms along the common boundary;
- **identities**: cylinders $\Sigma \times [0,1]$.

Disjoint union $\sqcup$ makes Bord_n a symmetric monoidal category with unit the empty $(n-1)$-manifold $\varnothing$.

Definition 5.2 (TQFT, Atiyah's axioms)**.** An *n-dimensional TQFT* valued in Vect_k (finite-dimensional k-vector spaces) is a symmetric monoidal functor $Z \colon \mathrm{Bord}_n \to \mathrm{Vect}_k$. Explicitly:

(Z1) Z assigns to each closed $(n-1)$-manifold Σ a finite-dimensional vector space $Z(\Sigma)$, with $Z(\varnothing) = k$;

(Z2) $Z(\Sigma_0 \sqcup \Sigma_1) \cong Z(\Sigma_0) \otimes Z(\Sigma_1)$ (monoidality);

(Z3) Z assigns to a cobordism $W\colon \Sigma_0 \to \Sigma_1$ a linear map $Z(W)\colon Z(\Sigma_0) \to Z(\Sigma_1)$, with $Z(W' \circ W) = Z(W') \circ Z(W)$ (gluing) and $Z(\Sigma \times [0,1]) = \mathrm{id}_{Z(\Sigma)}$;

(Z4) $Z(\overline{\Sigma}) \cong Z(\Sigma)^*$ (orientation reversal is dual).

A closed n-manifold M is a cobordism $\varnothing \to \varnothing$, so $Z(M) \in \mathrm{Hom}(k,k) = k$ is a numerical invariant (a partition function).

5.2 TQFT is a representation entry

Theorem 5.3 (TQFT realizes the Part I pipeline; monoidality is the Decomposition axiom)**.** *An n-dimensional TQFT $Z\colon \mathrm{Bord}_n \to \mathrm{Vect}_k$ is precisely a status-S representation entry $E = (\mathrm{Bord}_n, \mathrm{Vect}_k, Z, \mathsf{S})$ in the sense of Definition 2.2, realizing the pipeline of Definition 2.3 with*

$$\Phi = \textit{(inclusion of a closed } (n-1)\textit{-manifold as an object)},$$
$$\mathrm{Real}_\alpha = Z, \qquad \mathrm{Obs} = \dim \ (\textit{or trace/partition function}).$$

Moreover the symmetric monoidal structure of Z is exactly the Decomposition axiom (Definition 2.5) applied to disjoint unions and gluing:

$$Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2), \qquad Z(W_2 \circ W_1) = Z(W_2) \circ Z(W_1). \tag{5}$$

Proof. The data of Definition 2.2 are: a mathematical structure $M = \mathrm{Bord}_n \in \mathsf{Math}_{\mathrm{Man}}$, a physical target $P = \mathrm{Vect}_k \in \mathsf{Phys}$ (state spaces and linear processes), a translation datum $\tau = Z$, and a status σ. Since Z is an *actual functor* (Atiyah's axioms), τ is functorial and the entry is status-S by Remark 2.6. To exhibit the pipeline, note Φ includes an $(n-1)$-manifold Σ as an object of Bord_n; $\mathrm{Real}_\alpha = Z$ sends it to $Z(\Sigma) \in \mathrm{Vect}_k$; and $\mathrm{Obs} = \dim$ (or, on closed M, the scalar $Z(M)$) extracts a number. Each is functorial and they compose, so $\mathrm{Obs}_\alpha = \mathrm{Obs} \circ Z \circ \Phi$ is a realization pipeline (Definition 2.3).

Finally, (5) is the statement that Z is symmetric monoidal (Z2) and a functor (Z3). Reading disjoint union as "parallel composition of systems" and gluing as "sequential composition of processes," (5) says the observable of a composite decomposes into the observables of its pieces via $\otimes$ and $\circ$; this is precisely the Decomposition axiom instantiated on cobordisms. Uniqueness of the entry follows because a symmetric monoidal functor is determined by its action on objects and generating morphisms, which is the data specified. $\square$

Example 5.4 (A 2d TQFT is a commutative Frobenius algebra)**.** For $n = 2$, the classification theorem states that 2-dimensional TQFTs $Z\colon \mathrm{Bord}_2 \to \mathrm{Vect}_k$ correspond bijectively to commutative Frobenius algebras $A = Z(S^1)$: the pair of pants (a cobordism $S^1 \sqcup S^1 \to S^1$) gives the multiplication $A \otimes A \to A$, the reversed pair of pants gives the comultiplication, the disk ($\varnothing \to S^1$) gives the unit $\eta\colon k \to A$, and the reversed disk ($S^1 \to \varnothing$) gives the counit $\varepsilon\colon A \to k$ (the Frobenius trace form). Gluing the two disks yields the sphere S^2 ($\varnothing \to \varnothing$), which therefore evaluates to the scalar $\varepsilon(\eta(1)) = \varepsilon(1_A)$; the closed torus T^2, obtained instead as the trace of id_A (composition of coevaluation and evaluation), evaluates to $\dim_k A$. Associativity, commutativity, and the Frobenius relation are each a diffeomorphism between glued surfaces, hence forced by functoriality (5). This is the cleanest illustration that "the algebra of observables = the decomposition law of cobordisms." Status S.

5.3 The cobordism hypothesis

Theorem 5.5 (Cobordism hypothesis; Baez–Dolan, Lurie)**.** *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category with duals. The evaluation-at-a-point map*

$$\mathrm{Fun}^{\otimes}\big(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}\big) \xrightarrow{Z \mapsto Z(\mathrm{pt})} \mathcal{C}^{\mathrm{fd}}$$

is an equivalence between the space of fully extended framed n-dimensional TQFTs valued in $\mathcal{C}$ and the maximal ∞-groupoid $\mathcal{C}^{\mathrm{fd}}$ of fully dualizable *objects of $\mathcal{C}$. Thus a fully extended TQFT is determined, up to a contractible space of choices, by a single fully dualizable object $Z(\mathrm{pt})$. Status* **S** *(theorem of Lurie; proof sketched at length in [2]).*

Proof idea (cited, not reproduced). The full proof [2] is an induction on codimension. The key input is that $\mathrm{Bord}_n^{\mathrm{fr}}$ is the *free* symmetric monoidal (∞, n)-category with duals on a single object pt: every cobordism is generated, under composition and duality, by the point together with its (higher) evaluation and coevaluation morphisms. A symmetric monoidal functor out of a free such category is determined by the image of the generator, subject to that image being fully dualizable (so that all the duality data have targets). Hence $Z \mapsto Z(\mathrm{pt})$ is an equivalence onto $\mathcal{C}^{\mathrm{fd}}$. Framings can be traded for tangential G-structures by taking homotopy fixed points of an $O(n)$-action, giving the structured variants. $\square$

Corollary 5.6 (Rigidity of topological realization channels)**.** *In the extended setting, the topological realization channel is* rigid*: once the point-realization $Z(\mathrm{pt})$ is fixed as a fully dualizable object, all higher-codimension observables (line, surface, and defect data) are uniquely determined. Physically, fixing how the elementary local excitation realizes fixes the entire extended field theory. This is the strongest compositionality statement in the ladder and directly motivates Part V's abstraction of $\mathcal{C}$ into the general theory of (higher) categories.*

Proof. Immediate from Theorem 5.5: the evaluation map is an equivalence, so its inverse reconstructs Z (all its values in every codimension) from $Z(\mathrm{pt})$ functorially and essentially uniquely. $\square$

The functorial picture is summarized by the commuting triangle of monoidal functors, in which Z factors the Part I pipeline through the topological channel:

$$\begin{tikzcd} \mathrm{Bord}_n \arrow[r, "Z"] \arrow[dr, "\mathrm{Obs}_\alpha"'] & \mathrm{Vect}_k \arrow[d, "\dim / \mathrm{tr}"] \\ & k \end{tikzcd} \qquad \mathrm{pt} \xmapsto{Z} Z(\mathrm{pt}) \in \mathcal{C}^{\mathrm{fd}}$$

6 Dijkgraaf–Witten theory: locality from group cohomology

Dijkgraaf–Witten (DW) theory is a concrete, computable TQFT built from a finite group and a group cocycle. It illustrates the Locality/descent axiom cleanly, together with the coboundary = gauge reading of the dictionary.

Definition 6.1 (Group cohomology and DW data)**.** For a finite group G and coefficients $\mathbb{R}/\mathbb{Z}$ (trivial action), the *group cohomology* $H^\bullet(G;\mathbb{R}/\mathbb{Z}) := H^\bullet(BG;\mathbb{R}/\mathbb{Z})$ is computed by the bar complex: n-cochains are functions $\varphi\colon G^n \to \mathbb{R}/\mathbb{Z}$, with *additive* coboundary

$$(\delta\varphi)(g_1,\dots,g_{n+1}) = \varphi(g_2,\dots,g_{n+1}) \\ + \sum_{i=1}^{n}(-1)^i\,\varphi(g_1,\dots,g_ig_{i+1},\dots,g_{n+1}) + (-1)^{n+1}\varphi(g_1,\dots,g_n).$$

Here the leading term $\varphi(g_2,\dots,g_{n+1})$ carries no prefactor precisely because the G-action on the coefficients $\mathbb{R}/\mathbb{Z}$ is *trivial*; for a nontrivial action it would instead read $g_1{\cdot}\varphi(g_2,\dots,g_{n+1})$. A *Dijkgraaf–Witten datum* in dimension 3 is a class $\omega \in H^3(G;\mathbb{R}/\mathbb{Z})$; the associated phase $e^{2\pi i\,\omega} \in \mathrm{U}(1)$ recovers the equivalent multiplicative $\mathrm{U}(1)$-valued convention.

Theorem 6.2 (DW theory instantiates the Locality/descent axiom)**.** *Let G be finite and $\omega \in H^3(BG;\mathbb{R}/\mathbb{Z})$ a 3-cocycle. The state sum*

$$Z_\omega(X) = \frac{1}{|G|}\sum_{\phi\in\mathrm{Hom}(\pi_1(X),G)} \exp\bigl(2\pi i\,\langle\phi^*\omega,[X]\rangle\bigr) \tag{6}$$

over a closed oriented 3-manifold X defines a TQFT $Z_\omega\colon \mathrm{Bord}_3 \to \mathrm{Vect}_{\mathbb{C}}$; here the pairing $\langle\phi^*\omega,[X]\rangle$ lies in $\mathbb{R}/\mathbb{Z}$, so the exponential $e^{2\pi i(-)} \in \mathrm{U}(1)$ is well typed. The state sum is independent of the triangulation (hence well defined and local in the sense of Definition 2.4) precisely because ω is a cocycle; and a coboundary shift $\omega \mapsto \omega + \delta\eta$ changes Z_ω by nothing, matching the "coboundary = pure gauge" dictionary entry.

Proof. Present X by a triangulation T with an ordering of vertices; a discrete G-connection is a labelling of edges by G compatible on faces, i.e. a map $\phi\colon \pi_1(X) \to G$ up to gauge. Each tetrahedron contributes a phase $\exp\bigl(\pm 2\pi i\,\omega(g,h,k)\bigr)$ according to its orientation, and the state sum (6) is the product of these phases over tetrahedra, averaged over connections. Two triangulations of X are related by a finite sequence of *Pachner (bistellar) moves*. The invariance of the local weight under each Pachner move is *equivalent* to the cocycle condition $\delta\omega = 0$: the 2–3 Pachner move is exactly the (additive) pentagon/cocycle identity

$$\omega(h,k,l) - \omega(gh,k,l) + \omega(g,hk,l) - \omega(g,h,kl) + \omega(g,h,k) = 0 \quad \text{in } \mathbb{R}/\mathbb{Z}.$$

Hence $Z_\omega(X)$ is independent of T, establishing locality and gluing. If $\omega' = \omega + \delta\eta$ for a 2-cochain η, the extra phases are associated to faces and their exponents telescope to 0 (mod $\mathbb{Z}$) over the closed manifold X (each internal face is shared by two tetrahedra with opposite orientation), so $Z_{\omega'}(X) = Z_\omega(X)$. Functoriality on cobordisms follows because the state sum on a manifold with boundary defines a vector in the state space of the boundary and gluing corresponds to contracting these vectors. □

Remark 6.3 (Descent and the Equivalence axiom). Theorem 6.2 realizes the dashed arrow of the Part I master diagram in a discrete setting: the equivalence $\omega \sim \omega + \delta\eta$ (change of cocycle representative) is a gauge/equivalence relation under which the physical content Z_ω is invariant. Thus DW theory simultaneously discharges the Locality/descent axiom (triangulation independence) and the Equivalence axiom (coboundary invariance). Status **S**.

Example 6.4 ($G = \mathbb{Z}/N$, untwisted)**.** For $\omega = 0$, $Z_0(X) = |\operatorname{Hom}(\pi_1(X), \mathbb{Z}/N)|/N = |H^1(X; \mathbb{Z}/N)|/N$. On the 3-torus T^3 this counts flat $\mathbb{Z}/N$ connections, $|H^1(T^3; \mathbb{Z}/N)| = N^3$, so $Z_0(T^3) = N^2$; the ground-state degeneracy N^2 is the physical signature of $\mathbb{Z}/N$ topological order on the spatial torus, a status-**S** topological invariant recovered directly from H^1.

7 Bordism, anomalies, and symmetry-protected topological phases

The final layer of Part IV classifies *invertible* topological field theories, the mathematical home of anomalies and symmetry-protected topological (SPT) phases, by bordism-theoretic invariants.

Definition 7.1 (Bordism group)**.** For a tangential structure H (e.g. orientation, spin, or a map to BG), the *bordism group* Ω_n^H is the abelian group of closed n-manifolds with H-structure modulo the relation $M \sim M'$ iff $M \sqcup \overline{M'} = \partial W$ for some compact H-manifold W; addition is disjoint union. By the Pontryagin–Thom construction, $\Omega_n^H \cong \pi_n(MTH)$, the n-th stable homotopy group of the Thom spectrum MTH.

Definition 7.2 (Invertible field theory)**.** A field theory Z is *invertible* if $Z(\Sigma)$ is invertible in the target symmetric monoidal category for every Σ (e.g. a line in Vect) and each $Z(W)$ is an isomorphism. Invertible TQFTs are exactly the fully dualizable objects with trivial higher morphisms, hence factor through a spectrum-level map by Theorem 5.5.

Theorem 7.3 (Freed–Hopkins classification of invertible theories; SPT phases)**.** *Reflection-positive invertible n-dimensional field theories with tangential structure H are classified by the torsion subgroup of*

$$\left[MTH,\ \Sigma^{n+1} I\mathbb{Z}\right] \ \cong\ \left(I\mathbb{Z}\right)^{n+1}(MTH),$$

the degree-$(n+1)$ Anderson-dual cohomology of the Thom spectrum MTH (recall $E^k(X) = [X, \Sigma^k E]$, so an $(n+1)$-fold suspension lands in degree $n+1$) [7]. Consequently, a symmetry-protected topological phase with finite symmetry group G in d spacetime dimensions is classified, at the level of the relevant (co)bordism invariant, by a summand of $\Omega_d^H(BG)$ or its generalized-cohomology refinement. The pure-mathematics classification is status **S***; the identification of a* specific *lattice model's phase with a given bordism invariant is model-dependent, status* **S**/**H**.

Proof idea (cited). By Definition 7.2 an invertible theory is a map of spectra from the bordism spectrum MTH to the target; reflection positivity forces the target to be the Anderson dual $I\mathbb{Z}$ of the sphere (Freed–Hopkins [7], building on [6]). The classification is then the computation of the homotopy classes $[MTH, \Sigma^{n+1} I\mathbb{Z}]$, which by definition of Anderson duality is the stated generalized cohomology group. Coupling to a background G-symmetry replaces MTH by $MTH \wedge BG_+$, giving the $\Omega_d^H(BG)$-level statement. The physical identification with an SPT phase is via the partition function on mapping tori and its response to symmetry defects, which is model-dependent, hence the composite label. □

Remark 7.4 (Anomaly as failure of descent). An anomaly of an n-dimensional theory is precisely a nontrivial invertible $(n+1)$-dimensional theory in which it lives as a boundary — a failure of the partition function to be a number rather than a vector, i.e. a failure of the Locality/descent axiom (Definition 2.4) at the top codimension. Bordism invariants thus *measure* the obstruction to descent, closing the loop with Remark 3.18: anomalies are the top-dimensional analogue of the Mayer–Vietoris connecting map. Status **S**/**H**.

8 Results

We collect the principal results of Part IV and their status labels.

- **Homotopy invariance (Theorem 3.8)** and the **conserved global information functor** $H_\bullet$: hTop $\to$ GrAb (Definition 3.9): topological observables are conserved under continuous deformation. Status **S**; physical reading **S**/**H**.
- **Conservation principle (Theorem 3.10)**: over a contractible parameter base, (co)homology classes and their pairings are constant. Status **S**.
- **de Rham gauge picture (Proposition 3.15)** and **Poincaré duality (Theorem 3.19)**: closed forms are conserved fields, exact forms pure gauge, defects dual to field strengths. Status **S**; readings **S**/**H**.
- **Index theorem as pipeline identity (Theorem 4.6)**: the strongest status-**S** instance of observable = Real(abstract structure).
- **TQFT as representation entry (Theorem 5.3)**: the first functorial, status-**S** realization of the Part I pipeline; monoidality is the Decomposition axiom.
- **Cobordism hypothesis (Theorem 5.5, Corollary 5.6)**: topological realization channels are rigid; the bridge to Part V. Status **S**.
- **Dijkgraaf–Witten (Theorem 6.2)**: Locality/descent and Equivalence axioms from a group cocycle. Status **S**.
- **Bordism classification (Theorem 7.3)**: anomalies and SPT phases as bordism invariants; anomaly as failure of descent (Remark 7.4). Status **S** / **S**/**H**.

Proposition 8.1 (Status statistics of Part IV)**.** *Of the eight headline results above, seven are status-***S** *as pure mathematics (all but the model-specific half of Theorem 7.3); the heuristic content is confined to the* physical readings *of dictionary entries, and no result is status-***P***. Part IV is therefore the most rigorously anchored module of the ladder to date.*

Proof. By inspection of the individual status labels assigned at each theorem: TQFT functoriality (Theorem 5.3), the cobordism hypothesis (Theorem 5.5), the index theorem (Theorem 4.6), Dijkgraaf–Witten theory (Theorem 6.2), homotopy invariance (Theorem 3.8), the conservation principle (Theorem 3.10), and Poincaré duality (Theorem 3.19) are each established theorems of algebraic topology or mathematical physics; only the model-identification half of Theorem 7.3 is **S**/**H**, and the ontological framing (inherited, not asserted here) is the only place **P** occurs in the parent program. No headline result of Part IV is independently **P**. □

9 Companion formalization: Haskell and Lean

Following the parent program's formal-verification bridge ("math is code, code is math"; Curry–Howard–Lambek), Part IV ships companion code. The Haskell package compiles under GHC and encodes the structures of Sections 3 to 5; the Lean file is a best-effort sketch of the same signatures in a proof-assistant idiom.

9.1 Haskell: chain complexes, homology, and TQFT

The module `ChainComplex` realizes Definitions 3.1 and 3.3 over $\mathbb{Z}$-free chain groups with an explicit boundary matrix, computes Betti numbers via ranks (Smith-normal-form-free rational homology), and verifies $\partial^2 = 0$. The module `TQFT` encodes Definition 5.2 as a symmetric-monoidal-functor typeclass and instantiates the 2d Frobenius-algebra example (Example 5.4) and the untwisted Dijkgraaf–Witten count (Example 6.4). The following excerpt shows the core signatures:

```
data ChainComplex = CC { dims :: [Int], boundary :: [Matrix] }
d2isZero     :: ChainComplex -> Bool
bettiNumbers :: ChainComplex -> [Int]
class SymMonFunctor z where
  fUnit    :: z
  fTensor  :: z -> z -> z
  fCompose :: z -> z -> z
dwUntwisted  :: Int -> Int -> Double   -- (N, b_1 of manifold) -> Z(T^3)
```

The `Main` module runs all demonstrations: it verifies $\partial^2 = 0$ and prints Betti numbers for the circle, torus, and 2-sphere (recovering $b_\bullet(S^1) = (1,1)$, $b_\bullet(T^2) = (1,2,1)$, $b_\bullet(S^2) = (1,0,1)$), checks the Frobenius/TQFT gluing laws, and prints the Dijkgraaf–Witten degeneracy $Z_0(T^3) = N^2$ of Example 6.4.

9.2 Lean: chain-complex and homology types

The Lean sketch `AlgTop.lean` declares a `ChainComplex` structure with fields `d` (the boundary map in each degree) and a hypothesis `d_comp_d` asserting $\partial_n \circ \partial_{n+1} = 0$ for all n, defines cycles/boundaries/homology as subquotients, and states signatures for the `TQFT` structure (a symmetric monoidal functor $\mathrm{Bord}_n \to \mathrm{Vect}$) and for Theorem 4.6 and Theorem 6.2 as `theorem` statements with `sorry` placeholders. It is best-effort and not build-gated; its purpose is to record the intended machine-checkable interface, extending Part I's Lean skeleton to the topological domain.

10 Discussion

10.1 How Part IV builds on Part III

Part III supplies the geometric "spaces of physical possibility" (varieties, moduli, positive geometries). Part IV attaches to each such space its conserved global information: (co)homology, characteristic classes, and (via compactification and degeneration) a bordism class. The boundary stratification of a positive geometry (its "boundaries of boundaries") is literally a chain complex (Definition 3.1), and its homology measures which boundary charges are conserved. The variation-of-Hodge-structure and Gauss–Manin data of Part III are the smooth (de Rham) realization channel of the same cohomology treated here abstractly, so Parts III and IV are two realization channels (α = Hodge/de Rham vs. α = singular/topological) of one observable module, matched by the de Rham theorem.

10.2 How Part IV seeds Part V

TQFT (Theorem 5.3) is the first fully rigorous, status-**S** instance of the Part I pipeline in which the realization Real_α is a genuine functor. The cobordism hypothesis (Theorem 5.5) then shows the entire extended theory is generated by a single fully dualizable object in a symmetric monoidal (∞, n)-category $\mathcal{C}$. This forces the ambient grammar of the next module to be exactly the theory of (higher) categories with duality: Part V abstracts $\mathcal{C}$, functoriality, and the duality (evaluation / coevaluation) data into the general theory of categories, monoidal and dagger-compact structure, and homotopy type theory. In particular the "no natural diagonal" phenomenon behind no-cloning (Part V/VI) is the categorical shadow of the duality data that already organize Bord_n here.

10.3 Limitations

We reproduce and address the parent program's limitations as they bear on Part IV.

1. *Not every analogy is a theory.* The dictionary entries of Section A carry status labels; the **H** entries (e.g. "Postnikov tower = hierarchy of defects") are heuristic and are flagged as such, not asserted as established physics.

2. *No single universal functor* $\mathsf{Math} \to \mathsf{Phys}$. Part IV constructs *domain-local* functors ($H_\bullet$, Z), not one global functor; this is deliberate and consistent with the modular design.

3. *Model dependence of SPT identification.* Theorem 7.3 is status-**S** as a classification but only **S**/**H** when tied to a specific lattice model; each such claim needs its own citation.

4. *Extended/framed subtleties.* Theorem 5.5 is stated for framed bordism; structured (oriented, spin, G-) variants require homotopy fixed points of an $O(n)$-action and are more delicate. We flag but do not resolve these.

10.4 Connections to the synthesis backbone

Three items recur across modules and are load-bearing for the synthesis paper: (i) the realization pipeline (1), instantiated here as Z and as the index theorem; (ii) homology H_n as conserved cycles, which reappears in Part VI as surface-code logical operators $H_1(\Sigma; \mathbb{Z}/2)$; and (iii) the sheaf/stack (descent) dichotomy, which appears here as the Mayer–Vietoris connecting map and the anomaly-as-failed-descent principle (Remarks 3.18 and 7.4) and is made rigorous only in Part VI.

11 Conclusion

Part IV formalizes algebraic topology as the grammar of conserved global information within the Math→Physics representation library. We proved homotopy invariance and functoriality of homology and packaged them as a conserved global information functor $H_\bullet \colon \mathrm{hTop} \to \mathrm{GrAb}$; we read closed/exact forms as conserved fields/pure gauge and defects as dual field strengths; we reframed the Atiyah–Singer index theorem as an instance of observable = Real(abstract structure); and we exhibited TQFT as the first functorial, status-**S** realization of the Part I pipeline, with the cobordism hypothesis as its rigidity/classification capstone and Dijkgraaf–Witten theory as a computable model of the Locality and Equivalence axioms. The bordism classification of invertible theories placed anomalies and SPT phases into the same descent-theoretic frame.

The structural point of Part IV is that *functoriality becomes achievable in a genuinely topological, status-***S** *setting*. Every later appeal to "the realization functor" in the ladder is anchored by the concrete functor $Z \colon \mathrm{Bord}_n \to \mathrm{Vect}$ constructed here. This is precisely the bridge to Part V, which abstracts functoriality, monoidal structure, and duality into the universal grammar of categories and homotopy type theory. In the four-slogan compression of the parent program (mathematics is syntax, motives are semantics, periods are measurements, coalgebras are decomposition laws), Part IV supplies the missing structural clause: *topology is conservation, and functoriality is its law of composition.*

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A Algebraic topology dictionary (with status labels)

The following reproduces the algebraic-topology library with its epistemic status labels, verbatim in structure from the parent program's Appendix D. Composite labels are preserved.

Algebraic topology	**Physical representation**	**Semantic meaning**	**Status**
Topological space X	Configuration, spacetime, or state space	Domain of possibility	**S**
Point	State or event	Localized datum	**S**
Path	Evolution/worldline	Continuous transition	**S**
Loop	Closed evolution	Holonomy probe	**S**
Homotopy	Continuous deformation	Physically equivalent path	**S**
Homotopy class	Topological sector	Deformation-invariant class	**S**
$\pi_1(X)$	Winding structure	Monodromy, defect winding	**S**
$\pi_n(X)$	Higher holes	Brane/defect charge sectors	**S/H**
Homology $H_n(X)$	Cycles or holes	Conserved extended structures	**S**
Cohomology $H^n(X)$	Fields/flux observables	Charge detectors on cycles	**S**
Boundary operator ∂	Boundary of region	Where conservation appears	**S**
Cycle $\partial c = 0$	Closed conserved object	Conserved charge carrier	**S**
Boundary $c = \partial b$	Trivial conserved object	Pure gauge / null sector	**H**
Cocycle	Closed field/charge rule	Conservation condition	**S**
Coboundary	Gauge-exact field	Redundant potential	**S**
Long exact sequence	Boundary–bulk relation	Charge transfer	**S**
Mayer–Vietoris	Gluing local regions	Global from patches	**S**
Cup product	Product of classes	Interaction of observables	**S/H**
Cap product	Class $\frown$ cycle	Measurement over region	**S**
Poincaré duality	Cycles to fields	Defects as dual field strengths	**S**
de Rham cohomology	Forms mod exact	Gauge-invariant flux	**S**
Closed form	Conserved field	$d\omega = 0$ conservation	**S**
Exact form	Gauge-trivial field	$\omega = d\alpha$ redundancy	**S**
Hodge star $\star$	Dual field operation	EM/geometric duality	**S**

continued on next page

Algebraic topology dictionary (continued)

Algebraic topology	**Physical representation**	**Semantic meaning**	**Status**
Fiber bundle	Local d.o.f. over base	Internal state per point	**S**
Principal bundle	Gauge field topology	Gauge choices glued	**S**
Classifying space BG	Universal classifier	G-sector space	**S**
Characteristic class	Bundle invariant	Quantized charge/anomaly	**S**
Chern class	Complex bundle invariant	Monopole, quantized flux	**S**
Chern character	Charge map	Topological response/index	**S**
Stiefel–Whitney class	Real bundle obstruction	Orientation/spin obstruction	**S**
Spin structure	Lift of frame data	Fermion consistency	**S**
Index theorem	Topology to analysis	Zero modes/anomalies	**S**
K-theory	Cohomology of bundles	D-brane/phase charges	**S**
Spectra	Stable homotopy objects	Generalized charges	**S**
Generalized cohomology	Beyond ordinary charge	Exotic charges/phases	**S**
Steenrod operations	Cohomology operations	Constraints on phases	**S/H**
Postnikov tower	Layered homotopy data	Hierarchy of defects	**H**
Loop space ΩX	Space of closed paths	Recurrence dynamics	**S/H**
Suspension ΣX	Dimension shift	Dimensional lift of defects	**H**
Cobordism	Manifold as transition	Spacetime history	**S**
Bordism group	Manifolds mod boundary	Classification of phases	**S**
TQFT	Functor $\mathrm{Bord}_n \to \mathrm{Vect}$	Topology-to-algebra	**S**
Extended TQFT	Data in all codimensions	Particles, lines, surfaces	**S**
Fully dualizable object	Point-data of a TQFT	Local seed of global theory	**S**
Group cohomology	Cohomology of symmetry	Discrete action / SPT data	**S**
Dijkgraaf–Witten	Finite-group gauge theory	Action from group cocycle	**S**

References

[1] M. F. Atiyah, *Topological quantum field theories*, Publications Mathématiques de l'IHÉS **68** (1988), 175–186.

[2] J. Lurie, *On the classification of topological field theories*, in *Current Developments in Mathematics 2008*, Int. Press, 2009, 129–280; arXiv:0905.0465.

[3] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[4] J. Milnor and J. D. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies **76**, Princeton University Press, 1974.

[5] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990), 393–429.

[6] D. S. Freed, *Anomalies and invertible field theories*, Proc. Symp. Pure Math. **88** (2015), 25–45; arXiv:1404.7224.

[7] D. S. Freed and M. J. Hopkins, *Reflection positivity and invertible topological phases*, Geom. Topol. **25** (2021), 1165–1330; arXiv:1604.06527.

[8] J. C. Baez and J. Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995), 6073–6105.

[9] J. C. Baez and M. Stay, *Physics, topology, logic and computation: a Rosetta Stone*, in *New Structures for Physics*, Lecture Notes in Physics **813** (2010), 95–172; arXiv:0903.0340.

[10] M. F. Atiyah and I. M. Singer, *The index of elliptic operators I*, Annals of Mathematics **87** (1968), 484–530.

[11] A. Yu. Kitaev, *Fault-tolerant quantum computation by anyons*, Annals of Physics **303** (2003), 2–30; arXiv:quant-ph/9707021.

[12] D. Grady and D. Pavlov, *The geometric cobordism hypothesis*, arXiv:2111.01095 (2021).

[13] M. Kontsevich and D. Zagier, *Periods*, in *Mathematics Unlimited — 2001 and Beyond*, Springer, 2001, 771–808.