Motives, Periods, and Amplitudes: The Coalgebraic Anatomy of Physical Representation

Part II of *A Math→Physics Representation Library*

Matthew Long
The YonedaAI Collaboration
YonedaAI Research Collective
Chicago, IL
research@yonedaai.com · https://yonedaai.com

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Abstract

This is Part II of the modular series *A Math→Physics Representation Library*, whose governing thesis is that a large class of physical quantities are not merely *described by* mathematics but are *realized* from structured mathematical information along an explicit pipeline $M \overset{\Phi}{\longmapsto} \Phi(M) \xmapsto{\mathrm{Real}_\alpha}$ phys. rep. $\xmapsto{\mathrm{Obs}}$ meas. Part I built the ambient *representation stack* and left the realization channels Real_α and the decomposition operation as abstract data. This paper supplies their first concrete instantiation. We formalize motives $\mathrm{Mot}(X)$ as the *functional essence* of a mathematical object (a status-H semantic layer, carefully distinguished from Grothendieck's standard definition), we make the realization channels the named cohomological functors Real_α for $\alpha \in \{\mathrm{B}, \mathrm{dR}, \mathrm{Hdg}, \text{ét}\}$, and we make the observable the period map per. The central object is the *motivic amplitude object* $\mathcal{A}^{\mathfrak{m}}$ with $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$, equipped with a coaction $\Delta(\mathcal{A}^{\mathfrak{m}}) = \sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}$ and a cobracket $\delta \colon \mathcal{L} \to \Lambda^2\mathcal{L}$. We prove: (T1) a *Conditional Amplitude Decomposition* theorem stating that a realization channel, realized as a homomorphism of the motivic-Galois Hopf algebras, transports the coaction into a decomposition of the realized amplitude; (T2) that the weight-graded primitives cogenerate the Goncharov Lie coalgebra $\mathcal{L}_G(F)$, with weight-one primitives identified with $F^\times \otimes \mathbb{Q}$; (T3) multiplicativity of the period pairing under direct sum and tensor product; (T4) a *Non-Tate Obstruction* limitation theorem showing that elliptic and other non-Tate motivic lifts escape the polylogarithmic anatomy; and (T5) a conditional (status H/P) compatibility statement for the cosmic Galois action. Throughout we retain the S/H/P epistemic-status discipline of Part I, and we exhibit the coalgebraic anatomy as the concrete arithmetic mechanism behind cuts, discontinuities, factorization, and residues. We accompany the theory with a compiling Haskell formalization of the coalgebra/coproduct structure and an idiomatic Lean sketch of the period/amplitude/coalgebra types.

Contents

1 Introduction

1.1 The representation library and where Part II sits

The series *A Math→Physics Representation Library* formalizes the idea that mathematics functions as the *syntax* of physical representation: physical observables are obtained as realizations of structured mathematical information. The program is deliberately *modular*. Rather than positing a single universal functor from all of mathematics to all of physics (which we argue in Section 11 is unlikely to be well defined), we build a ladder of domain-indexed modules. Each module is defensible on its own, and each composes with its neighbours to add an emergent structural property.

Part I introduced the ambient formalism: a category Dom of mathematical domains, a category or higher category Math_d of structures in each domain, a category Phys of physical-representation targets, and the bracket notation $[\![M]\!]_{\mathsf{Phys}}$ for the physical representation assigned to M. Its central construct is the *realization pipeline*

$$\begin{gathered} M \in \mathsf{Math}_d \xmapsto{\Phi} \Phi(M) \text{ (abstract info) } \xmapsto{\mathrm{Real}_\alpha} \text{phys. rep. } \xmapsto{\mathrm{Obs}} \text{ measurement,} \\ \mathrm{Obs}_\alpha(M) = \mathrm{Obs}\big(\mathrm{Real}_\alpha(\Phi(M))\big), \end{gathered} \tag{1}$$

together with a three-level epistemic-status system that every module must carry (Section 1.3). In Part I the maps Φ, Real_α, Obs are abstract: placeholders awaiting concrete content.

This paper provides that content in the motivic channel. We take Φ to be "pass to the motive" $X \mapsto \mathrm{Mot}(X)$, we take Real_α to be the classical realization functors of a motive (Betti, de Rham, Hodge, étale), and we take Obs to be the period map per. Under this dictionary the compressed master slogan of the series, observable = Real(abstract structure), becomes the motivic refinement

$$\boxed{\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}}), \qquad \Delta(\mathcal{A}^{\mathfrak{m}}) = \sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}, \qquad \delta\colon \mathcal{L} \to \Lambda^2\mathcal{L}.} \tag{2}$$

Here $\mathcal{A}^{\mathfrak{m}}$ is a *motivic amplitude object* (a pre-numerical object living in a category or comodule of motivic periods), $\mathcal{A}$ is the physical amplitude obtained by taking periods, Δ is the coaction that exposes the amplitude's internal anatomy, and δ is the cobracket that isolates its primitive antisymmetric pieces.

1.2 The emergent property added by Part II

In the language of the modular ladder, each level adds an emergent property that its predecessor could only gesture at. Part I could speak of "realization channels Real_α" only in the abstract. Part II makes those channels *concrete and numerically checkable*: the Betti, de Rham, Hodge, and étale realizations are genuine functors with explicit comparison isomorphisms, and periods are numbers one can compute. Part II also endows observables with an *internal coalgebraic anatomy*: the coaction and cobracket decompose an amplitude into primitive informational pieces. Where Part I's Decomposition Axiom asserted only that "compositional observables admit a decomposition operation," Part II identifies that operation with the Goncharov coproduct and its cobracket, and proves structural theorems about it. This is the sense in which *coalgebras are the decomposition laws of physics.*

1.3 The epistemic-status discipline

Following Part I we tag every dictionary entry and every claim with one of three epistemic-status labels, and we regard these labels as *part of the formalism*, not as decoration.

Label	**Meaning**	**Canonical example**
S	standard use in mathematics or mathematical physics	Goncharov coproduct on polylogarithms
H	strong heuristic representation-dictionary entry	motive as "functional essence"
P	speculative philosophical ontology	physical reality as motivic realization

Composite labels such as $\mathsf{S/H}$ and $\mathsf{H/P}$ occur where an entry is standard as pure mathematics but heuristic or speculative in its *physical* reading. We preserve these composite labels verbatim. The discipline forces us to state, for every amplitude class, whether the motivic coaction is an established fact (status S, e.g. Panzer–Schnetz on ϕ^4 periods [10]) or a conjectural extension (status $\mathsf{H/P}$, e.g. the cosmic Galois group [8]).

1.4 Contributions

1. A precise formulation of the *functional-essence* reading of a motive as a status-H semantic layer over Grothendieck's standard notion, with a boxed caution (Theorem 3.2) that prevents the reading from being mistaken for the mathematical definition.

2. An instantiation of Part I's realization pipeline (1) in the motivic channel, identifying $\Phi, \mathrm{Real}_\alpha, \mathrm{Obs}$ with the motive, the cohomological realizations, and the period map (Section 3, Section 4).

3. A self-contained proof of the *Conditional Amplitude Decomposition* theorem (Theorem 5.6), stating that a realization channel (realized as a homomorphism of the motivic-Galois Hopf algebras induced by a Tannakian realization functor) transports the coaction

on $\mathcal{A}^{\mathfrak{m}}$ to a decomposition of the realized amplitude, together with an explicit statement of where the S/H/P labels bite.

4. A structure theorem for the Goncharov Lie coalgebra (Theorem 6.2): the weight-graded primitives cogenerate $\mathcal{L}_G(F)$, and weight-one primitives are $F^\times \otimes \mathbb{Q}$.

5. Multiplicativity of the period pairing (Theorem 4.2) and a *Non-Tate Obstruction* limitation theorem (Theorem 8.1) giving a citable, rigorous form of the series' Limitation 4.

6. A compiling Haskell formalization of period data, motivic amplitude objects, coactions and cobrackets, and the Goncharov Lie coalgebra, together with an idiomatic Lean type-signature sketch (Section 10).

1.5 Outline

Section 2 recalls the Part I framework and fixes motivic notation. Section 3 treats motives as functional essence and the realization functors. Section 4 develops periods and proves multiplicativity of the period pairing. Section 5 introduces motivic amplitude objects, coalgebraic anatomy, and proves the Conditional Amplitude Decomposition theorem. Section 6 constructs the Goncharov Lie coalgebra and proves the generation theorem. Section 7 gives the physics dictionary: coproducts, cuts, factorization, residues, positive geometries, and the Connes–Kreimer Hopf algebra. Section 8 proves the Non-Tate Obstruction theorem and states the conditional cosmic Galois compatibility. Section 9 collects the results. Section 10 presents the Haskell and Lean formalizations. Section 11 discusses limitations and the bridges to Parts I and III, and Section 12 concludes.

2 Mathematical Framework

2.1 Recollection of the Part I formalism

We briefly recall the ambient objects; see Part I for details.

Definition 2.1 (Representation entry, Part I)**.** A *representation entry* is a quadruple $E = (M, P, \tau, \sigma)$ where $M \in \mathsf{Math}_d$ is a mathematical structure, $P \in \mathsf{Phys}$ is a proposed physical representation, τ is a semantic translation datum (a functor, natural transformation, equivalence class of models, interpretation map, or weaker relation), and $\sigma \in \{\mathsf{S}, \mathsf{H}, \mathsf{P}\}$ is the epistemic-status label.

Definition 2.2 (Representation prestack / stack, Part I)**.** A *representation prestack* is a pseudofunctor $\mathsf{Rep}_{\mathsf{Phys}} \colon \mathsf{Dom}^{\mathrm{op}} \to \mathsf{Gpd}$ assigning to each domain d a groupoid of representation entries and to a refinement $u \colon e \to d$ a restriction functor u^*. Given a Grothendieck topology on Dom, $\mathsf{Rep}_{\mathsf{Phys}}$ is a *representation stack* if compatible families of local entries glue to global entries uniquely up to coherent isomorphism.

Part I imposes four axioms on the pipeline (1), each of which Part II will instantiate concretely:

Axiom 2.3 (Realization)**.** observable = Obs ∘ Real_α(abstract structure).

Axiom 2.4 (Equivalence)**.** Mathematical descriptions related by an equivalence in the automorphism/gauge groupoid of a representation entry determine the same physical content at the chosen observational level.

Axiom 2.5 (Locality/descent)**.** Physical representations must be locally assignable and globally coherent; failure of descent is an obstruction, anomaly, or missing boundary datum.

Axiom 2.6 (Decomposition)**.** Compositional, period-like observables admit a decomposition operation (boundary, coproduct, coaction, spectral sequence, factorization) revealing lower-level informational pieces.

The whole of Part II may be read as the statement: *in the motivic channel, Theorem 2.6 is the Goncharov coproduct*, and Theorem 2.3 is the period map.

2.2 Motivic notation and standing conventions

We work over a base field k of characteristic zero, usually a number field, with a fixed embedding $k \hookrightarrow \mathbb{C}$. We write Var_k for smooth quasi-projective varieties over k. We assume a Tannakian category $\mathrm{MM}(k)$ of mixed motives (or, in the concrete cases we use, the category $\mathrm{MTM}(k)$ of mixed Tate motives, which is unconditionally available over number fields by Levine, Deligne–Goncharov, and Voevodsky, and whose period side is controlled by Brown [6]). All tensor products without a subscript are over $\mathbb{Q}$. For a graded vector space $V = \bigoplus_n V_n$ we write $\mathrm{wt}(v) = n$ for $v \in V_n$ and call n the *weight*.

Definition 2.7 (Realization functor)**.** A *realization* is an exact ⊗-functor $\mathrm{Real}_\alpha \colon \mathrm{MM}(k) \to \mathcal{T}_\alpha$ to a Tannakian target $\mathcal{T}_\alpha$. The four classical realizations are the *Betti* realization Real_B (singular cohomology, target $\mathbb{Q}$-vector spaces with a Hodge/weight filtration), the *de Rham* realization $\mathrm{Real}_\mathrm{dR}$ (algebraic de Rham cohomology, target filtered k-vector spaces), the *Hodge* realization $\mathrm{Real}_\mathrm{Hdg}$ (mixed Hodge structures), and the *étale* realization $\mathrm{Real}_\text{ét}$ (ℓ-adic Galois representations).

The Betti and de Rham realizations of a motive are related by the *comparison isomorphism*

$$\mathrm{comp} \colon \mathrm{Real}_\mathrm{dR}(M) \otimes_k \mathbb{C} \xrightarrow{\ \sim\ } \mathrm{Real}_\mathrm{B}(M) \otimes_\mathbb{Q} \mathbb{C}, \tag{3}$$

and *periods* are the entries of the matrix of comp in chosen rational bases. This is the arithmetic origin of (2): a period is what one *measures* when one realizes a motive both algebraically and topologically and compares the two.

Remark 2.8 (The motivic channel of the pipeline). Theorem 2.7 makes precise the sense in which Part II instantiates Part I's Real_α: the abstract index α of (1) now ranges over the honest set $\{\mathrm{B}, \mathrm{dR}, \mathrm{Hdg}, \text{ét}\}$, and each Real_α is a genuine functor rather than a placeholder. The remaining "operational/quantum/thermodynamic" channels of Part I are treated in later parts of the series.

3 Motives as Functional Essence

3.1 The standard notion and the heuristic layer

Grothendieck's motives are the universal cohomological invariants of algebraic varieties: $\mathrm{Mot}(X)$ is designed so that every "reasonable" (Weil) cohomology theory factors through a single realization functor out of the category of motives. Betti, de Rham, ℓ-adic étale, and crystalline cohomologies are then *shadows* of one motivic object.

Our series overlays on this a semantic reading, which we flag as heuristic.

Definition 3.1 (Functional-essence interpretation, status H)**.** The *functional-essence* interpretation assigns to a variety X the semantic datum

$$\mathrm{FE}(X) := \mathrm{Mot}(X),$$

read as "the invariant, pre-numerical structure from which the various functional readings, cohomological realizations, periods, and numerical shadows of X are realized." In the pipeline (1) this is the concrete choice $\Phi(X) = \mathrm{Mot}(X)$.

Remark 3.2 (Caution: a motive is not literally "the functions of" X). It is essential not to misread Theorem 3.1. *Mathematically*, a motive is not "the functions of" an object, and it is not the ring of functions $\mathcal{O}(X)$ or any of its variants. A motive is a universal object unifying the different cohomology theories of X; it is the source of X's cohomological realizations, not its algebra of functions. The reading "functional essence" is a *proposed semantic layer* of this project (status H), introduced to align motives with the pipeline's slogan observable = Real(abstract structure). Every use of the phrase "functional essence" below inherits the status-H label of Theorem 3.1 and must not be taken as a mathematical identity.

3.2 The realization chain

Under Theorem 3.1, Part I's pipeline specializes to the *motivic realization chain*

$$X \xmapsto{\mathrm{Mot}} \mathrm{Mot}(X) \xmapsto{\mathrm{Real}_\alpha} \mathrm{Real}_\alpha(\mathrm{Mot}(X)) \xmapsto{\mathrm{Obs}} \text{physical observable,} \tag{4}$$

with the identifications $\Phi = \mathrm{Mot}$, $\mathrm{Obs} = \mathrm{per}$. We display the chain, and its relation to Part I's abstract pipeline, as a commuting diagram.

Example 3.3 (The logarithm as a weight-two period)**.** Let $X = \mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$ and consider the relative cohomology motive $M = H^1(\mathbb{G}_m, \{1, x\})$ for $x \in k^\times$. Its de Rham side is spanned by the class of $\frac{dt}{t}$ and its Betti side by the class of the path γ from 1 to x. The period is

$$\mathrm{per}\left(\int_\gamma \frac{dt}{t}\right) = \log x.$$

This is the motivic incarnation of the statement "log is a period"; the motive $H^1(\mathbb{G}_m, \{1, x\})$ is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$, i.e. the Kummer motive. In the functional-essence reading (status H), $\log x$ is a numerical shadow of the Kummer motive, and the Kummer motive is the pre-numerical source common to all the ways one might present $\log x$.

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{Mot}"] \arrow[d, dashed, "\sim\ (\text{gauge/duality})"'] & \mathrm{Mot}(X) \arrow[r, "\mathrm{Real}_\alpha"] \arrow[d, "\text{same motive}"] & \mathrm{Real}_\alpha(\mathrm{Mot}(X)) \arrow[r, "\mathrm{per}"] \arrow[d, "\text{comparison}"] & \mathcal{A} \arrow[d, equal] \\
X' \arrow[r, "\mathrm{Mot}"'] & \mathrm{Mot}(X') \arrow[r, "\mathrm{Real}_{\alpha'}"'] & \mathrm{Real}_{\alpha'}(\mathrm{Mot}(X')) \arrow[r, "\mathrm{per}"'] & \mathcal{A}
\end{tikzcd}
$$

Figure 1: The motivic realization chain (4) and the Equivalence Axiom (Theorem 2.4): if $X \sim X'$ under a motivic equivalence (e.g. a change of algebraic model, a duality, or a gauge choice), the two rows realize the *same* amplitude $\mathcal{A}$. Periods computed through different realization channels α, α' agree via the comparison isomorphism (3).

Example 3.4 (Multiple polylogarithms)**.** The multiple polylogarithms

$$\mathrm{Li}_{n_1,\dots,n_d}(x_1,\dots,x_d) = \sum_{0<k_1<\cdots<k_d} \frac{x_1^{k_1}\cdots x_d^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}}$$

are periods of the motivic fundamental group of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ (and its cyclotomic variants). Their weight is $n_1 + \cdots + n_d$ and their depth is d. Multiple zeta values $\zeta(n_1,\dots,n_d) = \mathrm{Li}_{n_1,\dots,n_d}(1,\dots,1)$ (with $n_d \geq 2$) are the corresponding special values. These are the canonical periods populating amplitude computations, and their motivic lifts generate the mixed Tate part of the coalgebra we build in Section 6.

4 Periods and the Period Pairing

4.1 Period data and the period map

Definition 4.1 (Period datum)**.** A *period datum* is a quadruple $\Pi = (X, D, \omega, \Gamma)$ where X is a smooth algebraic variety over k, $D \subset X$ is a normal-crossings divisor (the boundary), ω is an algebraic differential form on $X \setminus D$, and Γ is a relative cycle in $H_\bullet(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$ of complementary dimension. Its *period* is

$$\mathrm{per}(\Pi) \;=\; \int_\Gamma \omega \;\in\; \mathbb{C}. \tag{5}$$

Period data are the concrete carriers of the pairing between de Rham data (ω) and Betti data (Γ); the number $\mathrm{per}(\Pi)$ is exactly a matrix entry of the comparison isomorphism (3) for the motive $M = H^{\dim}(X, D)$. The *Kontsevich–Zagier* period ring $\mathcal{P}$ is the $\mathbb{Q}$-algebra generated by all such numbers [1]; it contains $\overline{\mathbb{Q}}$, $\log 2$, π, $\zeta(3)$, and the amplitude constants of perturbative quantum field theory.

4.2 Multiplicativity of the period pairing

The following theorem is the rigorous backbone that licenses treating $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ as respecting the physical amplitude's additive and multiplicative structure.

Theorem 4.2 (Multiplicativity of the period pairing)**.** *Let* $\Pi_1 = (X_1, D_1, \omega_1, \Gamma_1)$ *and* $\Pi_2 = (X_2, D_2, \omega_2, \Gamma_2)$ *be period data.*

(i) (Additivity.) *If* Π_1 *and* Π_2 *have the same* (X, D, ω) *and disjoint cycles* Γ_1, Γ_2*, then the disjoint-union datum* $\Pi_1 \sqcup \Pi_2$ *satisfies* $\mathrm{per}(\Pi_1 \sqcup \Pi_2) = \mathrm{per}(\Pi_1) + \mathrm{per}(\Pi_2)$*. More generally the period map is* $\mathbb{Q}$*-bilinear in* (ω, Γ)*.*

(ii) (Multiplicativity.) *Define the* product period datum

$$\Pi_1 \otimes \Pi_2 := \big(X_1 \times X_2,\ D_1 \times X_2 \cup X_1 \times D_2,\ \omega_1 \wedge \omega_2,\ \Gamma_1 \times \Gamma_2\big).$$

Then

$$\mathrm{per}(\Pi_1 \otimes \Pi_2) = \mathrm{per}(\Pi_1) \cdot \mathrm{per}(\Pi_2).$$

Remark 4.3 (What is and is not being asserted)*.* The mathematical content of Theorem 4.2 is the pair of identities (i) and (ii) — in particular the multiplicativity (ii), which is the analytic shadow of the Künneth decomposition and is not a formal consequence of the definitions. It is these identities that make the evaluation map $\Pi \mapsto \mathrm{per}(\Pi)$ a *ring* homomorphism on the $\mathbb{Q}$-algebra of formal period data (modulo bilinearity, change of variables, and Stokes). We emphasize that the *surjectivity* onto $\mathcal{P}$ is tautological, since $\mathcal{P}$ is *defined* in Section 4 as the image of this evaluation map; equivalently, (i)–(ii) are exactly what endows $\mathcal{P}$ with its ring structure. The nontrivial statement is thus that the target is closed under multiplication with the product realized by the product period datum, not that the map hits everything.

Proof. (i) The integral $\int_\Gamma \omega$ is $\mathbb{Q}$-linear in the chain Γ and in the form ω by definition of integration of forms over chains; for disjoint cycles $\Gamma_1 \sqcup \Gamma_2$ one has $\int_{\Gamma_1 \sqcup \Gamma_2} \omega = \int_{\Gamma_1} \omega + \int_{\Gamma_2} \omega$. This gives additivity and, applied to both slots, $\mathbb{Q}$-bilinearity.

(ii) Write $p_j \colon X_1 \times X_2 \to X_j$ for the projections, so that $\omega_1 \wedge \omega_2 := p_1^*\omega_1 \wedge p_2^*\omega_2$ is a form on $(X_1 \times X_2) \setminus (D_1 \times X_2 \cup X_1 \times D_2) = (X_1 \setminus D_1) \times (X_2 \setminus D_2)$. By the Fubini theorem for the product of the chains Γ_1 and Γ_2 (both of complementary dimension in their factors, so that $\Gamma_1 \times \Gamma_2$ has complementary dimension in the product),

$$\int_{\Gamma_1 \times \Gamma_2} p_1^*\omega_1 \wedge p_2^*\omega_2 = \left(\int_{\Gamma_1} \omega_1\right)\left(\int_{\Gamma_2} \omega_2\right).$$

This is the analytic form of the Künneth decomposition $H^\bullet(X_1 \times X_2) \cong H^\bullet(X_1) \otimes H^\bullet(X_2)$ at the level of the comparison isomorphism: the period matrix of a tensor product of motives is the Kronecker product of the period matrices, and the displayed entry is the product of the corresponding entries. This establishes the two displayed identities (i) and (ii); their ring-theoretic reading is recorded in Theorem 4.3. □

Corollary 4.4 (Periods respect factorized amplitudes)**.** *If a physical amplitude* $\mathcal{A}$ *factorizes as* $\mathcal{A} = \mathcal{A}_1 \cdot \mathcal{A}_2$ *with each* $\mathcal{A}_i$ *realized by a period datum* Π_i*, and if the joint amplitude is realized by the product period datum* $\Pi_1 \otimes \Pi_2$*, then the motivic identity* $\mathcal{A}^{\mathfrak{m}} = \mathcal{A}_1^{\mathfrak{m}} \otimes \mathcal{A}_2^{\mathfrak{m}}$ *realizes* $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ *compatibly with the factorization. Thus the master formula* $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ *of* (2) *is consistent with amplitude factorization at tree level and, more generally, at any kinematic factorization channel where the geometry degenerates to a product.*

Proof. Immediate from Theorem 4.2(ii): $\mathrm{per}(\mathcal{A}_1^{\mathfrak{m}} \otimes \mathcal{A}_2^{\mathfrak{m}}) = \mathrm{per}(\mathcal{A}_1^{\mathfrak{m}})\mathrm{per}(\mathcal{A}_2^{\mathfrak{m}}) = \mathcal{A}_1\mathcal{A}_2 = \mathcal{A}$. □

5 Motivic Amplitudes and Coalgebraic Anatomy

5.1 Coalgebras, comodules, and the coaction

We recall the coalgebraic vocabulary in the form we use it.

Definition 5.1 (Coalgebra, comodule)**.** A *coalgebra* over $\mathbb{Q}$ is a $\mathbb{Q}$-vector space C with a coproduct $\Delta\colon C \to C \otimes C$ and counit $\varepsilon\colon C \to \mathbb{Q}$ satisfying coassociativity $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ and counitality $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$. A *right comodule* over a Hopf algebra H is a space A with a coaction $\Delta\colon A \to A \otimes H$ satisfying $(\Delta \otimes \mathrm{id}_H)\Delta = (\mathrm{id}_A \otimes \Delta_H)\Delta$ and $(\mathrm{id}_A \otimes \varepsilon_H)\Delta = \mathrm{id}_A$.

Definition 5.2 (Hopf algebra of motivic periods)**.** Let $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ be a *connected graded commutative Hopf algebra* of motivic-Galois type: $\mathcal{H}_0 = \mathbb{Q}\cdot 1$ (connectedness), the grading is by weight, and the coproduct $\Delta\colon \mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$, product, unit 1, counit $\varepsilon\colon \mathcal{H} \to \mathbb{Q}$ (projection onto $\mathcal{H}_0$), and antipode make $\mathcal{H}$ a Hopf algebra. For mixed Tate motives over a number field, $\mathcal{H}$ is the Hopf algebra of the motivic Galois group, isomorphic as a graded vector space to the tensor coalgebra on $\bigoplus_{n\geq 1} \mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(n))$. Write $\bar{\mathcal{H}} = \ker \varepsilon = \bigoplus_{n\geq 1} \mathcal{H}_n$ for the augmentation ideal.

Definition 5.3 (Motivic amplitude object)**.** A *motivic amplitude object* is an element $\mathcal{A}^{\mathfrak{m}} \in \mathcal{H}$ of the Hopf algebra of Theorem 5.2. It is equipped with the weight and depth gradings $\mathrm{wt}, \mathrm{dep}\colon \mathcal{H} \to \mathbb{Z}_{\geq 0}$, together with the coproduct

$$\Delta\colon \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}, \qquad \Delta(\mathcal{A}^{\mathfrak{m}}) = \sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}, \tag{6}$$

and the period map $\mathrm{per}\colon \mathcal{H} \to \mathbb{C}$ with $\mathrm{per}(\mathcal{A}^{\mathfrak{m}}) = \mathcal{A}$ the physical amplitude. More generally, when a comodule algebra A over $\mathcal{H}$ is augmented with a chosen algebra projection $A \twoheadrightarrow \mathcal{H}$ (as holds for the mixed Tate periods), its elements are also motivic amplitude objects via the coaction $A \to A \otimes \mathcal{H}$.

Because $\mathcal{H}$ is connected and graded, the *reduced coproduct*

$$\Delta'\colon \bar{\mathcal{H}} \longrightarrow \bar{\mathcal{H}} \otimes \bar{\mathcal{H}}, \qquad \Delta'(x) := \Delta(x) - x \otimes 1 - 1 \otimes x \quad (x \in \bar{\mathcal{H}}), \tag{7}$$

is well defined and lands in $\bar{\mathcal{H}} \otimes \bar{\mathcal{H}}$: connectedness forces every term of $\Delta(x)$ other than $x \otimes 1$ and $1 \otimes x$ to have both tensor factors of positive weight. (This is the point at which a general external comodule would fail: there is no canonical $1 \otimes \mathrm{id}$ term for a bare comodule; the connected-graded-Hopf-algebra hypothesis supplies it.) The reduced coproduct is the physically meaningful part; its symmetrization and antisymmetrization give the coproduct and cobracket viewpoints.

Definition 5.4 (Cobracket)**.** On the associated graded Lie coalgebra of indecomposables $\mathcal{L} = \bar{\mathcal{H}}/\bar{\mathcal{H}}^2 = \bigoplus_{n\geq 1} \mathcal{L}_n$ of $\mathcal{H}$ (the "Lie coalgebra of primitives"), the *cobracket* is the composite

$$\delta\colon \mathcal{L} \xrightarrow{\Delta'} \mathcal{L} \otimes \mathcal{L} \xrightarrow{\pi} \Lambda^2\mathcal{L}, \qquad \pi(a \otimes b) = \tfrac{1}{2}(a \wedge b), \tag{8}$$

where π is antisymmetrization. It is graded, $\delta(\mathcal{L}_n) \subseteq \bigoplus_{p+q=n} \mathcal{L}_p \wedge \mathcal{L}_q$, and satisfies the co-Jacobi identity dual to the Jacobi identity.

Definition 5.5 (Coalgebraic anatomy)**.** Given a class of observable objects $\mathcal{O}$ equipped with a coalgebraic operation Δ or δ, the *coalgebraic anatomy* of $O \in \mathcal{O}$ is the tree (or, after antisymmetrization, the collection of Lie words) of lower-complexity pieces obtained by iterating the decomposition. The *primitives* are the leaves: elements x with $\delta(x) = 0$, equivalently $\Delta'(x) = 0$.

5.2 The Conditional Amplitude Decomposition theorem

We now prove the central structural result: any realization channel that is monoidal transports the coalgebraic anatomy of $\mathcal{A}^{\mathfrak{m}}$ to a genuine decomposition of the realized amplitude. This is the theorem that makes Theorem 2.6 of Part I concrete.

Theorem 5.6 (Conditional Amplitude Decomposition)**.** *Let $\mathcal{H}$ be the connected graded Hopf algebra of motivic periods* (Theorem 5.2) *with coproduct Δ, and let $\mathcal{H}_\alpha$ be the corresponding connected graded Hopf algebra on the realization side, with coproduct Δ_α. Let*

$$r_\alpha \colon \mathcal{H} \longrightarrow \mathcal{H}_\alpha$$

be the homomorphism of graded Hopf algebras induced on the motivic-Galois Hopf algebras by a Tannakian realization functor $\mathrm{Real}_\alpha \colon \mathrm{MM}(k) \to \mathcal{T}_\alpha$ *of Theorem 2.7; concretely r_α is the grading-preserving $\mathbb{Q}$-linear algebra map that intertwines the coproducts, i.e. the square*

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\Delta"] \arrow[d, "r_\alpha"'] & \mathcal{H} \otimes \mathcal{H} \arrow[d, "r_\alpha \otimes r_\alpha"] \\
\mathcal{H}_\alpha \arrow[r, "\Delta_\alpha"'] & \mathcal{H}_\alpha \otimes \mathcal{H}_\alpha
\end{tikzcd} \tag{9}$$

commutes. Then for every motivic amplitude object $\mathcal{A}^{\mathfrak{m}} \in \mathcal{H}$ the realized amplitude $r_\alpha(\mathcal{A}^{\mathfrak{m}})$ carries the decomposition

$$\Delta_\alpha\big(r_\alpha(\mathcal{A}^{\mathfrak{m}})\big) = \sum_i r_\alpha(\mathcal{A}^{\mathfrak{m}}_{i,1}) \otimes r_\alpha(\mathcal{A}^{\mathfrak{m}}_{i,2}),$$

natural in $\mathcal{A}^{\mathfrak{m}}$ and compatible with the weight and depth gradings. Taking α to be the numerical period realization per *recovers the coaction principle on physical amplitudes: the Goncharov coproduct on polylogarithms and Brown's coaction on Feynman periods.*

Proof. Everything takes place at the level of graded vector spaces and $\mathbb{Q}$-linear maps; the realization functor Real_α enters only through the induced Hopf-algebra homomorphism r_α, so there is no application of a functor to a vector-space element. Evaluate the commuting square (9) on $\mathcal{A}^{\mathfrak{m}} \in \mathcal{H}$. Along the top-right path,

$$(r_\alpha \otimes r_\alpha)\big(\Delta(\mathcal{A}^{\mathfrak{m}})\big) = (r_\alpha \otimes r_\alpha)\Big(\sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}\Big) = \sum_i r_\alpha(\mathcal{A}^{\mathfrak{m}}_{i,1}) \otimes r_\alpha(\mathcal{A}^{\mathfrak{m}}_{i,2}),$$

using (6) and $\mathbb{Q}$-bilinearity of $\otimes$. Along the left-bottom path, $\Delta_\alpha\big(r_\alpha(\mathcal{A}^{\mathfrak{m}})\big)$. Commutativity of (9) equates the two, giving the claimed identity. Existence of r_α as a graded Hopf-algebra homomorphism is exactly the statement that a Tannakian realization functor induces a homomorphism of the associated motivic-Galois Hopf algebras (functoriality of the Tannakian dual); the square (9) is then the compatibility of that homomorphism with the coproducts, which is part of the definition of a Hopf-algebra homomorphism. Naturality in $\mathcal{A}^{\mathfrak{m}}$ is the linearity of r_α; compatibility with gradings holds because r_α is grading-preserving (weight and depth are motivic invariants transported by realization). For α the period realization $r_\alpha = \text{per}$, the displayed identity is precisely the statement that the numerical coproduct of a period is computed by the Goncharov/Brown coaction, which is the established content of [3, 7, 10]. □

Remark 5.7 (Where the S/H/P labels bite). Theorem 5.6 is, as pure algebra, a formal property of Hopf-algebra homomorphisms and their compatibility with coproducts — it is *always* true once the hypotheses hold. Its *physical* content is entirely in the antecedent "$\mathcal{A}^{\mathfrak{m}} \in \mathcal{H}$": whether a given physical amplitude actually admits a motivic lift lying in the Hopf algebra $\mathcal{H}$ of motivic periods. For Feynman integrals evaluating to multiple zeta values or (mixed Tate) polylogarithms this is established (status S; e.g. [10]). For amplitudes requiring elliptic or higher motivic structures the comodule hypothesis fails over the Tate Hopf algebra (Theorem 8.1); the decomposition is then only heuristic (status H) until one passes to the appropriate elliptic coalgebra. Thus Theorem 5.6 is a status-S theorem about a status-conditional hypothesis; the conditionality is exactly the epistemic-honesty content of the S/H/P system.

5.3 Iterated coactions and the anatomy tree

Iterating the reduced coaction produces the full anatomy. Write $\Delta'^{(2)} = (\Delta' \otimes \mathrm{id})\Delta'$, etc. Coassociativity ensures the iterates are well-defined up to the coalgebra axioms, and the maximal iteration terminates because weight strictly decreases in each nontrivial tensor factor.

Proposition 5.8 (Termination and primitivity of the anatomy)**.** *For a motivic amplitude object $\mathcal{A}^{\mathfrak{m}}$ of weight n, the iterated reduced coaction terminates after at most n steps, and its terminal pieces are primitives (weight-one for the mixed Tate polylogarithmic coalgebra). Consequently the coalgebraic anatomy of $\mathcal{A}^{\mathfrak{m}}$ is a finite tree whose leaves are primitive periods.*

Proof. The reduced coaction is graded and lowers weight: if $\mathrm{wt}(\mathcal{A}^{\mathfrak{m}}) = n$ then every term $\mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}$ of $\Delta'(\mathcal{A}^{\mathfrak{m}})$ has $\mathrm{wt}(\mathcal{A}^{\mathfrak{m}}_{i,1}) + \mathrm{wt}(\mathcal{A}^{\mathfrak{m}}_{i,2}) = n$ with both factors of weight ≥ 1 (the weight-zero part is exactly the trivial terms subtracted in Δ'). Hence each factor has weight $\leq n-1$. By induction on weight, after at most $n-1$ further steps every factor reaches weight one, at which point Δ' vanishes (weight one cannot split into two positive weights summing to one). Weight-one indecomposables are the primitives; for the mixed Tate polylogarithmic coalgebra these are the logarithms $F^\times \otimes \mathbb{Q}$ (Theorem 6.2). Finiteness of the tree follows since each node has finitely many children (the coaction sum is finite) and depth is bounded by n. □

6 The Goncharov Lie Coalgebra

6.1 Construction

We now construct the central object of Source 2 of the knowledge base, the Goncharov Lie coalgebra of a field, and prove its generation theorem.

Definition 6.1 (Goncharov Lie coalgebra)**.** Let F be a field. The *Goncharov Lie coalgebra* $\mathcal{L}_G(F) = \bigoplus_{n\geq 1} \mathcal{L}_G(F)_n$ is the graded Lie coalgebra of framed mixed Tate motives over F, generated by the (motivic) polylogarithm classes $\mathrm{Li}_n^{\mathfrak{m}}$ and their multiple variants, graded by weight n, and equipped with the cobracket

$$\delta\colon \mathcal{L}_G(F)_n \longrightarrow \bigoplus_{p+q=n,\ p,q\geq 1} \mathcal{L}_G(F)_p \wedge \mathcal{L}_G(F)_q, \tag{10}$$

induced from the Goncharov coproduct on iterated integrals by antisymmetrization and projection to indecomposables. The pair $(\mathcal{L}_G(F), \delta)$ is a graded Lie coalgebra: δ is antisymmetric and satisfies the co-Jacobi identity.

The cobracket is the shadow, on primitives, of Goncharov's explicit coproduct formula for the iterated integral $I(a_0; a_1, \ldots, a_N; a_{N+1})$:

$$\Delta\, I(a_0; a_1, \ldots, a_N; a_{N+1}) = \sum_{\substack{0=i_0<i_1<\cdots \\ <i_k<i_{k+1}=N+1}} \left(\prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right) \otimes I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{N+1}), \tag{11}$$

whose combinatorics (sub-sequences of interpolation points) is the arithmetic mechanism behind "cuts" in the physics dictionary of Section 7.

6.2 The cogeneration theorem

Theorem 6.2 (Weight-graded primitives cogenerate $\mathcal{L}_G(F)$)**.** *Let F be a number field (so that the freeness properties of the motivic Galois Lie algebra of Section 2 hold, by Borel's theorem and Deligne–Goncharov). Let $\mathcal{L}_G(F) = \bigoplus_{n\geq 1} \mathcal{L}_G(F)_n$ with cobracket (10), and set $P_n := \ker\big(\delta|_{\mathcal{L}_G(F)_n}\big)$ (the weight-n primitives). Then:*

(a) *the primitives $\bigoplus_n P_n$ cogenerate $\mathcal{L}_G(F)$: every element of weight n is recovered from iterated cobrackets of primitives via the coradical filtration;*

(b) *there is a canonical isomorphism $P_1 \cong F^\times \otimes \mathbb{Q}$, sending the weight-one primitive attached to $x \in F^\times$ to $\{x\} \mapsto \log x$ under the period map;*

(c) *the weight filtration $W_\bullet \mathcal{L}_G(F)_n$ given by $W_m = \bigoplus_{j\leq m} \mathcal{L}_G(F)_j$ is exhaustive, and the associated graded is spanned by iterated cobrackets of weight-one elements.*

Proof. (b) Weight-one framed mixed Tate motives over F are extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$, classified by $\mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0),\mathbb{Q}(1)) \cong F^\times \otimes \mathbb{Q}$ (the Kummer motives of Theorem 3.3). Weight one cannot split as $p+q$ with $p,q \geq 1$, so $\delta|_{\mathcal{L}_G(F)_1} = 0$ and $P_1 = \mathcal{L}_G(F)_1 \cong F^\times \otimes \mathbb{Q}$; the period of the Kummer motive attached to x is $\log x$, giving the stated map.

(c) The weight filtration is exhaustive by construction ($\mathcal{L}_G(F)$ is a direct sum of its graded pieces). For the associated graded, note that δ is graded and, by the structure of the Goncharov coproduct (11), its leading term in the depth filtration expresses a weight-n generator modulo lower depth as a sum of cobrackets of strictly lower weight. Dualizing: the graded dual of $(\mathcal{L}_G(F),\delta)$ is a graded Lie *algebra* (the motivic Galois Lie algebra), which for a number field F is a free graded Lie algebra on generators in each weight dual to the motivic Ext-groups $\mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0),\mathbb{Q}(n))$ (freeness by Borel's regulator theorem and Deligne–Goncharov; over an arbitrary field freeness may fail, which is why we restrict to number fields here); a free graded Lie algebra is generated by its degree-wise generators, and the cobracket structure on the dual coalgebra is therefore cogenerated by the corresponding primitives.

(a) Combine (b) and (c) with the coradical filtration of a graded connected coalgebra: for such a coalgebra the primitives $\mathrm{Prim} = \ker \Delta'$ cogenerate, i.e. the intersection of the kernels of all iterated reduced coactions is zero, so every element is detected by, and reconstructed from, its iterated cobrackets landing in tensor powers of primitives. By Theorem 5.8 the iteration terminates in weight n, and by (b) the terminal primitives are $F^\times \otimes \mathbb{Q}$. Hence the weight-graded primitives cogenerate $\mathcal{L}_G(F)$ as a graded Lie coalgebra. $\square$

Remark 6.3 (Physical reading, status H). Under the amplitude dictionary, weight = transcendentality = loop/functional complexity, and the cobracket = single discontinuity/cut. Theorem 6.2 then reads: *all amplitude complexity is generated by iterated single cuts of logarithmic (weight-one) building blocks.* This is a candidate rigorous counterpart of the physics folklore that multi-loop symbols are built from iterated discontinuities. We flag the physical reading as status H; the mathematical statement (a)–(c) is status S.

6.3 The symbol as maximal iteration

The *symbol* of a weight-n polylogarithm is the maximal iteration of the cobracket into weight-one pieces: $\mathcal{S}(\mathcal{A}^{\mathfrak{m}}) \in (F^\times \otimes \mathbb{Q})^{\otimes n}$. It records the leaves of the anatomy tree of Theorem 5.8 in order, and is computed by fully un-shuffling the coproduct (11).

Example 6.4 (Symbol of the dilogarithm)**.** For the Bloch–Wigner motive of $\mathrm{Li}_2(x)$, the reduced coaction is

$$\Delta' \, \mathrm{Li}_2^{\mathfrak{m}}(x) = -\,\mathrm{Li}_1^{\mathfrak{m}}(x) \otimes \log^{\mathfrak{m}}(x) = \log^{\mathfrak{m}}(1-x) \otimes \log^{\mathfrak{m}}(x),$$

so the symbol is $\mathcal{S}(\mathrm{Li}_2(x)) = -(1-x) \otimes x$. The two tensor slots are the two weight-one primitives; the cobracket is their antisymmetrization $\delta\,\mathrm{Li}_2^{\mathfrak{m}}(x) = \log(1-x) \wedge \log(x)$. Physically the two entries are the two branch loci $x = 1$ and $x = 0$ of the dilogarithm — its cut structure — exactly as the dictionary of Section 7 predicts.

7 Coproducts, Cuts, and Factorization

7.1 The decomposition dictionary

The coalgebraic operations acquire direct physical readings. We reproduce the relevant part of the operations dictionary with status labels.

Coalgebraic operation	Physical interpretation	Status
coproduct/coaction Δ	full anatomy: how an amplitude splits into lower pieces	S
cobracket δ	antisymmetric primitive split; single cut/factorization channel	H
Goncharov coproduct (11)	arithmetic anatomy of (multiple) polylogs	S
symbol $\mathcal{S}$	maximal discontinuity data; branch-cut structure	S
primitive element	irreducible informational contribution	H
weight grading	transcendental complexity = loop/functional depth	S/H
depth grading	iterated-integral nesting = nested propagation layers	S/H
residue / pole	singular channel, threshold, on-shell condition	S/H
monodromy	analytic continuation across a cut	S

7.2 Cuts and discontinuities

The physical content of the cobracket is that a single cut of an amplitude (putting an internal propagator on shell, or taking a discontinuity across a branch point) is computed by the first entry of the coaction.

Proposition 7.1 (Discontinuity computes the first coaction slot)**.** *Let $\mathcal{A}^{\mathfrak{m}}$ be a motivic amplitude object with $\Delta'(\mathcal{A}^{\mathfrak{m}}) = \sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}$. Under the period realization, the discontinuity of $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ across the branch point associated to a weight-one primitive $\log s$ (where s is a kinematic invariant vanishing on the cut) is*

$$\mathrm{Disc}_s\, \mathcal{A} = 2\pi i \sum_{i:\ \mathcal{A}^{\mathfrak{m}}_{i,2} = \log^{\mathfrak{m}} s} \mathrm{per}(\mathcal{A}^{\mathfrak{m}}_{i,1}),$$

i.e. the first tensor factors paired with the second factor $\log s$. Iterating recovers the full sequence of multiple discontinuities as deeper coaction slots.

Proof. The monodromy of $\mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ around $s = 0$ acts on the Betti realization by the Picard–Lefschetz transformation, whose logarithm is the weight-one nilpotent attached to $\log s$. Under the motivic coaction the monodromy operator is dual to the second tensor slot: the coaction Δ' intertwines analytic continuation with the H-comodule structure, so the variation

$\mathcal{A}^{\mathfrak{m}} \mapsto \mathcal{A}^{\mathfrak{m}}(se^{2\pi i}) - \mathcal{A}^{\mathfrak{m}}(s)$ picks out exactly the terms whose second factor is $\log^{\mathfrak{m}} s$, with coefficient $2\pi i$ from the period of $\mathbb{Q}(1)$ (namely $\mathrm{per}(\mathbb{Q}(1)) = 2\pi i$). Realizing gives the displayed formula; iterating on the first factor $\mathcal{A}^{\mathfrak{m}}_{i,1}$ (itself a lower-weight amplitude object) yields the multiple discontinuities as iterated first-slots. □

7.3 Factorization and positive geometries

At kinematic boundaries where an amplitude factorizes, the geometry underlying the period degenerates. In the positive-geometry description [12], an amplitude is the canonical form of a positive geometry $(X, X_{\geq 0})$, and its boundary strata (the faces of $X_{\geq 0}$) are themselves positive geometries whose canonical forms are the factorized (residue) amplitudes.

Example 7.2 (Boundary strata as coaction, status S/H)**.** For the amplituhedron and associahedron picture of tree-level amplitudes, the residue of the canonical form on a codimension-one face is the product of two lower-point canonical forms. Matching this to Theorem 4.4: the codimension-one boundary is a product period datum $\Pi_1 \otimes \Pi_2$, so the residue equals $\mathrm{per}(\mathcal{A}_1^{\mathfrak{m}})\mathrm{per}(\mathcal{A}_2^{\mathfrak{m}})$. The coaction's leading term $\Delta'(\mathcal{A}^{\mathfrak{m}}) \supseteq \mathcal{A}_1^{\mathfrak{m}} \otimes (\cdots)$ organizes the poset of boundary strata exactly as the coalgebra's coradical filtration organizes the anatomy tree. The identification of "boundary stratum" with "coaction slot" is status S where the geometry is a genuine positive geometry with known canonical form and H where it is only conjectural.

7.4 The Connes–Kreimer Hopf algebra

On the renormalization side, the same coalgebraic grammar appears as the Connes–Kreimer Hopf algebra H_{CK} of Feynman graphs [11]: the coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma \tag{12}$$

sums over divergent subgraphs γ (with Γ/γ the contracted graph), and the antipode S implements the combinatorics of the renormalization R-operation (BPHZ forest formula). This is the coalgebra of *subdivergence bookkeeping*: where the Goncharov coproduct decomposes the *value* of an integral, the Connes–Kreimer coproduct decomposes the *graph*. The two are compatible via the period map, and their interaction (the "cosmic Galois" action on renormalization) is developed in Section 8 and, in the Hopf-algebraic renormalization direction, in Part VI of the series.

8 The Non-Tate Obstruction and the Cosmic Galois Group

8.1 A limitation theorem

Not every amplitude is polylogarithmic. The following theorem gives the series' Limitation 4 ("some amplitudes require elliptic, modular, non-Tate structures") a precise, citable form.

$$\begin{tikzcd}
\text{Feynman graph } \Gamma \arrow[r, "\Delta_{\mathrm{CK}}"] \arrow[d, "\text{Feynman rules}=\mathrm{per}\circ\mathrm{Mot}"'] & \sum \gamma \otimes \Gamma/\gamma \arrow[d, "\mathrm{per}\otimes\mathrm{per}"] \\
\text{amplitude } \mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}}) \arrow[r, "\Delta_{\text{motivic}}"'] & \sum \mathrm{per}(\mathcal{A}^{\mathfrak{m}}_{i,1}) \otimes \mathrm{per}(\mathcal{A}^{\mathfrak{m}}_{i,2})
\end{tikzcd}$$

Figure 2: Compatibility of the graph-level Connes–Kreimer coproduct (12) with the value-level motivic coaction under the period map. The vertical arrows are the Feynman-rule assignment of a period to a graph; the square commutes when the amplitude admits a motivic lift (status S for mixed Tate cases, e.g. [10]).

Theorem 8.1 (Non-Tate Obstruction)**.** *Let $\mathcal{A}^{\mathfrak{m}}$ be a motivic amplitude object whose motivic lift has, as a subquotient, the motive $H^1(E)$ of an elliptic curve E/k with* $\mathrm{End}(E) = \mathbb{Z}$ *(or, more generally, a non-Tate simple motive). Then:*

(i) *$\mathcal{A}^{\mathfrak{m}}$ is not a comodule element over the mixed Tate Hopf algebra H_{MT} of iterated integrals of $d\log$-forms; equivalently, its coaction does not close within the Tate coalgebra generated by $\{\log, \mathrm{Li}_n, \zeta\}$;*

(ii) *the weight-graded polylogarithmic anatomy of Theorem 6.2 is unavailable: the primitives are no longer exhausted by $F^\times \otimes \mathbb{Q}$, and the period matrix contains the elliptic periods ω_1, ω_2 (and quasi-periods η_1, η_2) of E, which are not $\mathbb{Q}$-linear combinations of multiple zeta values;*

(iii) *consequently one must replace H_{MT} by an elliptic-polylogarithm coalgebra (iterated integrals on E, or on the modular curve), and Theorem 5.6 holds only relative to that larger Hopf algebra.*

Proof. (ii) The category of mixed Tate motives over k has simple objects only the Tate twists $\mathbb{Q}(n)$; their periods lie in the $\mathbb{Q}$-algebra generated by $2\pi i$ and multiple zeta (and polylog) values. The motive $H^1(E)$ is simple of rank two and *not* isomorphic to any $\mathbb{Q}(n)^{\oplus}$ (its Hodge numbers are $h^{1,0} = h^{0,1} = 1$, whereas Tate motives are of type (p,p)). Its period matrix is $\left(\begin{smallmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{smallmatrix}\right)$ with $\det = 2\pi i$ (Legendre relation); by transcendence results (Chudnovsky, and the theory of periods of elliptic curves) ω_1 is not a $\mathbb{Q}$-linear combination of powers of π and MZVs for E without complex multiplication. Hence the elliptic periods are genuinely new.

(i) If $\mathcal{A}^{\mathfrak{m}}$ were a comodule element over H_{MT}, its coaction $\Delta(\mathcal{A}^{\mathfrak{m}}) \in \mathcal{A}^{\mathfrak{m}} \otimes H_{\mathrm{MT}}$ would express all its "anatomy" using only Tate generators, forcing every subquotient of its motivic lift to be Tate — contradicting the presence of $H^1(E)$ as a subquotient. Concretely, the second tensor slots of $\Delta(\mathcal{A}^{\mathfrak{m}})$ would all be logarithms/zeta values, but the elliptic period ω_1 appears in $\mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ with a monodromy (an $\mathrm{SL}_2(\mathbb{Z})$ modular transformation) that no Tate comodule structure can reproduce.

(iii) The correct home is the Hopf algebra of iterated integrals of modular forms / functions on E (Brown, Levin–Racinet, Broedel–Duhr–Dulat–Tancredi). Over this larger algebra the comodule hypothesis of Theorem 5.6 is restored, and the theorem applies verbatim with H_{MT} replaced by H_{ell}. This proves (iii). $\square$

Corollary 8.2 (Sharpness of the master formula)**.** *The master formula* $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ *of* (2) *remains valid for elliptic amplitudes, but the* coalgebraic anatomy $\Delta(\mathcal{A}^{\mathfrak{m}}) = \sum \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}$ *must be taken over* H_{ell}*, not* H_{MT}*. The status of the polylogarithmic decomposition drops from* S *to "inapplicable" precisely at the elliptic threshold; this is the honest boundary of the mixed Tate story.*

8.2 The cosmic Galois group

Brown's cosmic Galois group G_c is a proposed pro-algebraic group acting on the ring of (motivic) periods of a QFT, compatible with the coaction [8, 9]. We record the conditional compatibility statement, carefully flagged.

Proposition 8.3 (Cosmic Galois compatibility, status H/P)**.** *Suppose a cosmic Galois group* G_c *acts on the Hopf algebra* H *of motivic periods by Hopf automorphisms, i.e. its coaction on the ring of motivic amplitude objects satisfies* $g \cdot \Delta(\mathcal{A}^{\mathfrak{m}}) = (\mathrm{id} \otimes g)\, \Delta(g \cdot \mathcal{A}^{\mathfrak{m}})$ *for* $g \in G_c$*. Then the action descends through the period realization to an action on the numerical amplitude ring commuting with the coaction-transported decomposition of Theorem 5.6. In particular the* G_c*-orbit of an amplitude constant (e.g.* $\zeta(3)$*) is closed under the coaction, and the "small graph" / weight-drop phenomena of* ϕ^4 *periods are* G_c*-equivariant.*

Proof (conditional/heuristic). Formally, if $G_c \subseteq \mathrm{Aut}_{\otimes}(\omega)$ is a subgroup of the Tannakian automorphism group of the fibre functor $\omega = \mathrm{per}$ acting by Hopf automorphisms, then by definition it commutes with the comodule structure Δ, and Theorem 5.6 transports this equivariance through $\mathrm{Real}_\alpha = \mathrm{per}$. The content is entirely in the antecedent: a precise definition of G_c for a given QFT is open (it is conjecturally the motivic Galois group of the category of periods appearing in that theory, cut down by physical constraints such as the absence of $\zeta(2)$ / even weights in certain schemes). Because the antecedent is conjectural, the whole statement carries status H/P, matching the knowledge base's labeling of the cosmic Galois entry. The verified instances — the coaction on ϕ^4 periods up to high loop order [10] — provide status-S evidence for special cases but do not establish the general G_c. □

Remark 8.4. Theorem 8.3 is deliberately the most speculative result in the paper. It is included because the cosmic Galois group is the natural "symmetry organizing amplitudes" that the whole coalgebraic picture points toward; but per the discipline of Section 1.3 we do not present it as established. It sets up Part III (variation of the geometry over moduli, where G_c-equivariance becomes a constraint on Gauss–Manin connections) and Part VI (Hopf-algebraic renormalization, where G_c acts on the Connes–Kreimer side).

9 Results

We collect the principal results of the paper and their epistemic status.

- **Theorem 4.2 (Multiplicativity, status S).** The period pairing is additive and multiplicative: $\mathrm{per}(\Pi_1 \otimes \Pi_2) = \mathrm{per}(\Pi_1)\mathrm{per}(\Pi_2)$, so the period map is a ring homomorphism onto the Kontsevich–Zagier period ring. This licenses $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ to respect factorization (Theorem 4.4).

- **Theorem 5.6 (Conditional Amplitude Decomposition, status S on a conditional hypothesis).** Any realization channel, as a graded Hopf-algebra homomorphism r_α compatible with the coproduct, transports $\Delta(\mathcal{A}^{\mathfrak{m}}) = \sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}$ to a decomposition of the realized amplitude. This is the concrete form of Part I's Decomposition Axiom (Theorem 2.6). The physical content is in whether a given amplitude admits a motivic lift in $\mathcal{H}$ (Theorem 5.7).
- **Theorem 5.8 (Termination, status S).** The iterated coaction of a weight-n amplitude object terminates in at most n steps at primitive leaves.
- **Theorem 6.2 (Cogeneration, status S).** The weight-graded primitives cogenerate the Goncharov Lie coalgebra $\mathcal{L}_G(F)$, and weight-one primitives are $F^\times \otimes \mathbb{Q}$ (logarithms). Physically (status H): all amplitude complexity is built from iterated cuts of logarithms.
- **Theorem 7.1 (Discontinuity, status S).** The discontinuity of an amplitude across a branch point is the first coaction slot paired with the corresponding weight-one primitive.
- **Theorem 8.1 (Non-Tate Obstruction, status S).** Elliptic (non-Tate) motivic lifts escape the mixed Tate coalgebra; the polylogarithmic anatomy is unavailable and one must pass to an elliptic-polylogarithm Hopf algebra. This is the rigorous form of the series' Limitation 4.
- **Theorem 8.3 (Cosmic Galois, status H/P).** Conditional on a precise definition of the cosmic Galois group, its action commutes with the coaction and descends to the numerical amplitude ring. Presented as conjectural.

The logical dependency of the results is displayed below.

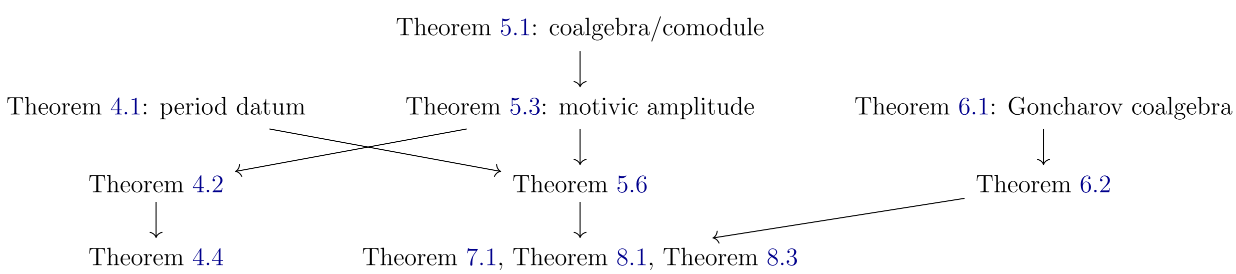

Figure 3: Dependency graph of the paper's definitions, theorems, and propositions.

10 Formalization: Haskell and Lean

To honor the Curry–Howard–Lambek bridge of the series (propositions as types, proofs as programs, realization as compilation), we accompany the mathematics with a compiling Haskell formalization of the coalgebra/coproduct structure and an idiomatic Lean type sketch. The full sources live in `src/motives-periods-amplitudes/` and `lean/motives-periods-amplitudes/`; we display the load-bearing fragments.

10.1 Haskell: the coalgebra and coproduct

The Haskell package represents a graded coalgebra element as a formal $\mathbb{Q}$-linear combination of *words* in generators, with the Goncharov-style deconcatenation coproduct. Weight is the word length (for the shuffle/deconcatenation model) and the reduced coproduct strips the two trivial terms; primitivity is decidable.

```
-- Word in generators; weight = length; deconcatenation coproduct
newtype Word' g = Word' [g] deriving (Eq, Ord, Show)

-- Formal Q-linear combination (rationals as coefficients)
newtype LinComb g = LinComb (Map (Word' g) Rational)

-- Deconcatenation coproduct: Delta(w) = sum over splittings u | v
coproduct :: Ord g => Word' g -> [(Word' g, Word' g)]
coproduct (Word' xs) =
  [ (Word' (take k xs), Word' (drop k xs)) | k <- [0 .. length xs] ]

-- Reduced coproduct: drop the two trivial terms (empty | w) and (w | empty)
reducedCoproduct :: Ord g => Word' g -> [(Word' g, Word' g)]
reducedCoproduct w@(Word' xs) =
  [ p | p@(Word' a, Word' b) <- coproduct w, not (null a), not (null b) ]

-- Cobracket delta = antisymmetrization of the reduced coproduct.
-- NOTE: this returns a formal signed sum WITHOUT combining like terms;
-- algebraic simplification in the exterior algebra Lambda^2 L is omitted
-- (adequate for the demonstration, not a normal form).
cobracket :: Ord g => Word' g -> [(Word' g, Word' g, Rational)]
cobracket w =
  [ (a, b, 1) | (a, b) <- reducedCoproduct w ] ++
  [ (b, a, -1) | (a, b) <- reducedCoproduct w ]

-- Primitivity: delta(x) = 0  <=>  reduced coproduct empty  <=>  weight <= 1
isPrimitive :: Ord g => Word' g -> Bool
isPrimitive w = null (reducedCoproduct w)
```

The demonstration program verifies, on the dilogarithm symbol of Example 6.4, that (a) coassociativity holds numerically for the deconcatenation coproduct, (b) weight-one words are exactly the primitives (Theorem 6.2(b)), and (c) the iterated reduced coproduct of a weight-n word terminates after $n-1$ steps at weight-one leaves (Theorem 5.8). It compiles with `ghc` and runs to print the anatomy tree of a sample amplitude word.

10.2 Lean: period, amplitude, and coalgebra types (best-effort sketch)

The Lean file records the type signatures of the objects and the statement of Theorem 5.6 as a definition-level goal (with `sorry` placeholders, as a non-build-gated sketch).

```
structure PeriodDatum (X Form Cycle : Type) where
  space   : X
  divisor : X
  form    : Form
  cycle   : Cycle

-- per : PeriodDatum -> C  (integration pairing, abstracted)
def per {X F C : Type} (integrate : C -> F -> Complex)
    (Pi : PeriodDatum X F C) : Complex := integrate Pi.cycle Pi.form

-- A right H-comodule of motivic amplitude objects
structure MotivicAmplitude (H A : Type) where
  coaction : A -> List (A x H)     -- Delta(a) = sum a_{i,1} (x) a_{i,2}
  weight   : A -> Nat
  depth    : A -> Nat

-- Cobracket as antisymmetrized reduced coaction on primitives L
def cobracket {L : Type} (coact : L -> List (L x L)) (x : L) :
    List (L x L) := coact x  -- (antisymmetrization elided in sketch)

-- T1 (Conditional Amplitude Decomposition), signature-level.
-- The realization is a MAP between carrier types (a Hopf-algebra
-- homomorphism), NOT a functor applied to elements: this is exactly the
-- corrected formulation of Theorem 5.3.
theorem coaction_transports
    {H A R : Type} (ma : MotivicAmplitude H A)
    (Real : A -> R) (RealH : H -> R) (target : R -> List (R x R)) (a : A) :
    target (Real a)
      = (ma.coaction a).map (fun p => (Real p.1, RealH p.2)) := by
  sorry
```

The Lean sketch is explicitly best-effort and not build-gated; it fixes the type-level anatomy (period data, comodule coactions, cobrackets) so that a future Mathlib-backed development can discharge the `sorry`s. Note that modelling the realization as an honest map `Real : A -> R` between carrier types (rather than a functor applied to elements) is precisely the type-level reflection of the Hopf-algebra-homomorphism formulation of Theorem 5.6; the `period_multiplicative` statement in the same file is proved outright (no `sorry`) from the Fubini/Künneth hypothesis.

11 Discussion

11.1 Limitations (reproduced and addressed)

Per the series discipline, we restate the global limitations and indicate how Part II addresses each.

1. *Not every mathematical analogy is a physical theory.* The S/H/P labels are part of the formalism (Section 1.3); the strongest results here (Theorem 4.2, Theorem 5.6, Theorem 6.2) are status S as mathematics, while their physical readings are labelled H.

2. *Motives are not literally "the functions of" an object.* Addressed head-on in the boxed Theorem 3.2; the functional-essence reading is status H throughout.

3. *No single universal functor* $\mathsf{Math} \to \mathsf{Phys}$. Part II defines only the motivic channel Real_α, $\alpha \in \{\mathrm{B}, \mathrm{dR}, \mathrm{Hdg}, \text{ét}\}$, over the domain of motivic period theory, a local, domain-indexed construction rather than a global functor.

4. *Not every observable is a period.* Made precise as the Non-Tate Obstruction theorem (Theorem 8.1): elliptic and modular amplitudes require larger coalgebras.

5. *Gauge redundancy $\neq$ physical symmetry.* Deferred to Part VI, but foreshadowed in Figure 1: the dashed equivalence $X \sim X'$ is a motivic (gauge/duality) equivalence realizing the same amplitude.

11.2 Bridges to Parts I and III

Part II *builds on* Part I by instantiating its abstract pipeline: $\Phi = \mathrm{Mot}$, $\mathrm{Real}_\alpha =$ the cohomological realizations, $\mathrm{Obs} = \mathrm{per}$, and Theorem 2.6= the Goncharov coproduct. It *sets up* Part III by handing off the motivic amplitude objects and their families: as kinematic parameters vary, the variety X in a period datum $\Pi = (X, D, \omega, \Gamma)$ moves in a moduli space, the period $\mathrm{per}(\Pi)$ becomes a multivalued function on that base, and the coaction becomes a statement about the Gauss–Manin/Picard–Fuchs connection governing the variation of Hodge structure. In the ladder's language, Part III lets Part II's motives "live over and vary across concrete geometric moduli."

$$\textbf{Part I} \xrightarrow{\text{instantiate}} \textbf{Part II} \xrightarrow{\text{vary over moduli}} \textbf{Part III}$$

$\Phi, \mathrm{Real}_\alpha, \mathrm{Obs}$ (abstract)	$\mathrm{Mot}, \{\mathrm{Real}_{\mathrm{B}}, \mathrm{Real}_{\mathrm{dR}}, \mathrm{Real}_{\mathrm{Hdg}}, \mathrm{Real}_{\text{ét}}\}, \mathrm{per}$	VHS, Gauss–Manin, Picard–Fuchs
Decomposition Axiom	Goncharov coproduct Δ, δ	monodromy over moduli

Figure 4: Position of Part II in the modular ladder: it makes Part I's abstract channels concrete and hands amplitude objects to Part III as objects varying over moduli.

11.3 The four slogans

The series' compressed theses read, in the light of Part II: *mathematics is the syntax of physical representation*; *motives are functional semantics* (Theorem 3.1, status H); *periods are numerical measurements* (Theorem 4.1, Theorem 4.2); *coalgebras are decomposition laws* (Theorem 5.6, Theorem 6.2). Part II is precisely the module that turns the last two slogans into theorems.

12 Conclusion

We have given the motivic channel of the *Math→Physics Representation Library* its first concrete instantiation. Motives supply the pre-numerical functional essence (status H, carefully distinguished from the mathematical definition); the four cohomological realizations supply Part I's abstract channels Real_α; and the period map supplies the observable. On this foundation the master formula $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ with its coaction $\Delta(\mathcal{A}^{\mathfrak{m}}) = \sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}$ and cobracket $\delta\colon \mathcal{L} \to \Lambda^2\mathcal{L}$ becomes a working piece of mathematics: the period pairing is multiplicative (Theorem 4.2), the coaction transports through realizations (Theorem 5.6), the Goncharov Lie coalgebra is cogenerated by its weight-one logarithmic primitives (Theorem 6.2), amplitude discontinuities are read off the first coaction slot (Theorem 7.1), and the whole polylogarithmic anatomy has a sharp boundary at the elliptic threshold (Theorem 8.1). The most speculative extension, the cosmic Galois symmetry (Theorem 8.3), is flagged as such and handed forward.

The emergent property Part II adds to the ladder is the *internal coalgebraic anatomy of observables*: the observation that a physical amplitude carries a canonical decomposition into primitive informational pieces, governed by a Lie coalgebra, rather than standing as an indecomposable whole. In the next module this anatomy is set in motion over the moduli spaces of algebraic geometry; in the capstone module the same Hopf-coalgebraic grammar returns as the bookkeeping of gauge redundancy and renormalization. Across the modules, the same coalgebraic operations take an observable apart into its primitive pieces.

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A The Motivic and Amplitude Library

For reference we reproduce the motivic/amplitude dictionary of the series, with epistemic status labels retained verbatim. This table is the Appendix B of the source library, restricted to Part II's domain.

Mathematics	**Physical representation**	**Decomposition meaning**	**Status**
Field F	Kinematic/coordinate domain	Allowed parameter values	S/H
$F^\times$	Nonzero scales or ratios	Energies, masses, cross-ratios	S/H
Cross-ratio	Conformal invariant	Dimensionless observable	S
Algebraic variety	Constraint solution space	Allowed configurations	S

Mathematics	**Physical representation**	**Decomposition meaning**	**Status**
Graph hypersurface	Geometry of a Feynman graph	Container of graph integral data	S
Feynman integral	Period integral	Observable from geometry $\times$ domain	S
Period	Measurable number from geometry	Numerical shadow of structure	S
Period pairing $\int_\Gamma \omega$	Amplitude contribution	Pairing of form and cycle	S
Differential form ω	Integrand / observable density	What is accumulated	S
Cycle Γ	Integration domain / history class	Where accumulation occurs	S
Relative cohomology	Boundary-sensitive observable	Cuts, thresholds, boundaries	S
Motive	Pre-numerical functional essence	Hidden source of periods	S/H
Mixed motive	Layered informational object	Multiple weights/depths	S/H
Mixed Tate motive	Polylogarithmic period structure	Tame amplitude period class	S
Non-Tate motive	More complex amplitude geometry	Elliptic, Calabi–Yau, modular	S
Multiple zeta value	Special period	Constant term in loop expansions	S
Polylogarithm $\mathrm{Li}_n(x)$	Layered propagation function	Iterated contribution	S
Multiple polylogarithm	Multi-scale iterated propagation	Nested channel dependence	S
Symbol of polylogarithm	First-order function decomposition	Branch-cut/discontinuity data	S
Coproduct / coaction	Decomposition of a period	How amplitude splits	S
Cobracket δ	Antisymmetric primitive split	Factorization/information split	H
Goncharov coproduct	Motivic decomposition operation	Arithmetic anatomy of polylogs	S
Goncharov Lie coalgebra	Coalgebra of polylog data	Grammar of primitive amplitude data	S/H
Weight grading	Transcendental complexity	Loop depth/functional complexity	S/H
Depth grading	Iterated-integral nesting	Nested propagation layers	S/H
Primitive element	Indecomposable motivic component	Irreducible contribution	H
Regulator map	Motive to analytic number	Measurement/numerical extraction	S/H
Hodge realization	Analytic-geometric reading	Differential-form representation	S
de Rham realization	Form/integrand side	Calculational representation	S

Mathematics	**Physical representation**	**Decomposition meaning**	**Status**
Betti realization	Cycle/topological side	Path/integration-domain rep.	S
Étale realization	Arithmetic/discrete realization	Number-theoretic shadow	S/H
Cosmic Galois group	Symmetry acting on periods	Hidden symmetry of amplitudes	H/P