Sheaves, Stacks, Gauge Redundancy, and Quantum High-Availability

Part VI of *A Math→Physics Representation Library*:
The Capstone Module — Descent, Isotropy, and Fault-Tolerant Encoding

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Abstract

This is Part VI, the capstone module, of the seven-part modular series *A Math→Physics Representation Library*. We formalize four intertwined mathematical domains as physical-representation modules: (i) **sheaves** as the theory of compatible local data glued along a Grothendieck topology; (ii) **stacks** as fibered categories in groupoids satisfying descent, the objects that remember not only local data and gluing but also the redundancy symmetries (automorphisms) of that data; (iii) **gauge redundancy**, the physical phenomenon in which many mathematical descriptions represent one physical content, realized mathematically as a quotient stack $[X/G]$; and (iv) **quantum high-availability**, the fault-tolerant encoding of logical information across redundant physical degrees of freedom. Our central structural thesis is that these four are one construction viewed through four lenses: *many local representatives, one invariant logical content, plus explicit data recording the redundancy.* We give rigorous definitions and prove four principal results. Theorem (T1) shows that the quotient stack retains isotropy, $\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x)$, and is therefore a strictly more faithful representation object than the coarse quotient X/G whenever automorphism-counting observables are present. Theorem (T2), a stackification/descent theorem, discharges the forward reference left open in Part I: the informally posited "representation stack" is literally recovered as the descent-theoretic stackification of the representation prestack, closing the ladder. Theorem (T3) proves that surface-code logical operators are the nontrivial homology classes $H_1(\Sigma; \mathbb{Z}_2)$ of the code surface, giving a literal (status S) instance of "gauge redundancy ↔ quantum high-availability" and reusing Part IV's homology dictionary verbatim. Theorem (T4) identifies the Connes–Kreimer antipode with the BPHZ counterterm via Birkhoff decomposition, exhibiting renormalization as Hopf-algebraic decomposition and reusing Part II's coalgebraic backbone. We then survey the holographic quantum-error-correction bridge (Almheiri–Dong–Harlow; Pastawski–Yoshida–Harlow–Preskill; Harlow) as a more rigorous, operator-algebraic strengthening of the heuristic slogan "gauge redundancy is to physics what fault-tolerant encoding is to quantum information." Throughout we preserve the series' three-level epistemic status system (S/H/P) and reproduce the series limitations, in particular the distinction between gauge redundancy and physical symmetry. Accompanying Haskell and Lean formalizations encode the sheaf/stack/stabilizer-code structures.

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1 Introduction

1.1 The modular series and where Part VI sits

The series *A Math→Physics Representation Library* advances a single organizing thesis: many physical quantities are *obtained as realizations* of structured mathematical information rather than merely *described by* it. The program is deliberately **modular**: rather than assert a single grand functor $\mathsf{Math} \to \mathsf{Phys}$, it builds a hierarchy of six standalone, formally verifiable modules, each introducing one mathematical domain as a physical-representation faculty and composing on the previous ones, followed by a seventh synthesis paper. The ladder is

$$\underbrace{\mathrm{I}}_{\substack{\text{repr.}\\ \text{stack}}} \to \underbrace{\mathrm{II}}_{\substack{\text{motives,}\\ \text{periods}}} \to \underbrace{\mathrm{III}}_{\substack{\text{alg.}\\ \text{geometry}}} \to \underbrace{\mathrm{IV}}_{\substack{\text{alg.}\\ \text{topology}}} \to \underbrace{\mathrm{V}}_{\substack{\text{cat. theory,}\\ \text{HoTT}}} \to \underbrace{\mathrm{VI}}_{\substack{\text{sheaves, stacks,}\\ \text{gauge, quantum HA}}} \to \underbrace{\text{Synth.}}_{\text{recomp.}}$$

The present paper is **Part VI**, the capstone before synthesis. Its job is threefold. First, it introduces the mathematics of *descent*: presheaves, sheaves, fibered categories, and stacks. Second, it uses that machinery to make rigorous the two constructions that the whole series has been circling: the *quotient stack* $[X/G]$ as the mathematical home of gauge redundancy, and the *stackification* that converts a prestack into a stack. Third, it exhibits the principal cross-module bridge in the series: gauge redundancy in physics and fault-tolerant encoding in quantum information are two readings of one structure, *many physical representatives, one logical/observable content, with explicit data recording the redundancy.*

1.2 The circular closure of the ladder

A structural fact drives the design. Part I *posits*, by analogy, that representation entries should glue "not by equality but by equivalence," i.e. that the assignment of physical representations to mathematical domains behaves like a *stack*. But Part I does not define a stack; it defers to a forward reference. Part VI *is* that forward reference resolved: here we define fibered categories, descent, and stackification, and prove (Theorem 5.1, T2) that the representation prestack of Part I admits a universal stackification, satisfying Part I's informal "representation stack" definition exactly. The ladder is therefore *literally circular*: Part VI supplies the rigorous foundation for Part I's opening formalism. We flag this as the candidate "closure theorem" for the synthesis paper (Part VII).

1.3 Contributions

1. A self-contained, arxiv-style formalization of sheaves and stacks over a site (Sections 3 and 4), with the physical-representation dictionary of the series attached to each construction and carrying $\mathsf{S}/\mathsf{H}/\mathsf{P}$ status labels.

2. **T1** (Theorem 4.4, Theorem 4.5): the quotient stack retains isotropy, $\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x)$, and is strictly more faithful than the coarse quotient for any automorphism-counting observable (orbifold Euler characteristic, $1/|\,\mathrm{Stab}\,|$ path-integral weights).

3. **T2** (Theorem 5.1): a descent/stackification theorem that discharges Part I's forward reference, closing the ladder.

4. **T3** (Theorem 7.2): surface-code logical operators are the classes of $H_1(\Sigma; \mathbb{Z}_2)$; the code space of a genus-g surface code is $(\mathbb{C}^2)^{\otimes 2g}$. This is a status-$\mathsf{S}$ bridge unifying Part IV (homology), Part VI (stacks/gauge), and quantum information.

5. **T4** (Theorem 8.3): the Connes–Kreimer antipode computes the BPHZ counterterm and renormalization is the Birkhoff decomposition of the Feynman-rules character, reusing Part II's Hopf-algebraic decomposition backbone.

6. A survey (Section 9) of holographic/operator-algebra quantum error correction as a rigorous strengthening of the series' central heuristic (status $\mathsf{H} \to$ partially S), with the epistemic status clearly labeled.

7. Companion Haskell (`src/sheaves-stacks-gauge-quantum-ha/`) and Lean (`lean/sheaves-stacks-gauge-quantum-ha/`) formalizations.

1.4 Notation and standing conventions

We work with categories, functors, and (2-)groupoids as developed in Part V. **Set** is the category of sets, **Grpd** the (2-)category of groupoids, **Cat** that of (small) categories, **Vect** that of vector spaces over a fixed field. For a category $\mathcal{C}$ and object c, $\mathrm{Aut}_{\mathcal{C}}(c)$ is the automorphism group of c. For a group G acting on a set/space X we write $\mathrm{Stab}_G(x) = \{g \in G : g \cdot x = x\}$ for the stabilizer, $\mathrm{Orb}_G(x) = \{g \cdot x : g \in G\}$ for the orbit, X/G for the coarse orbit set, and $[X/G]$ for the *quotient stack* (Theorem 4.3). Hilbert spaces are finite-dimensional unless stated otherwise; $\mathcal{P}_n$ is the n-qubit Pauli group. We reserve $\simeq$ for equivalence of groupoids/categories and $\cong$ for isomorphism of objects.

2 Mathematical framework: the representation stack recalled

We briefly recall the load-bearing structures of Parts I–V that Part VI builds on and discharges; this makes the paper self-contained while fixing notation for the closure theorem.

2.1 The realization pipeline and the master formulas

Definition 2.1 (Realization pipeline, after Part I)**.** Let Dom be a category of *mathematical domains*, Phys a category of *physical representations*. For a domain object M, the *realization pipeline* is the composite

$$\mathrm{Obs}_\alpha(M) \;=\; \mathrm{Obs}\Big(\mathrm{Real}_\alpha(\Phi(M))\Big), \tag{1}$$

where $\Phi(M)$ is the abstract informational content of M, Real_α is a choice of realization channel (indexed by α), and Obs extracts an observable. The compressed slogan is observable = Real(abstract structure).

Part VI instantiates (1) with Real_α the *quantum encoding* $\iota : \mathcal{H}_{\mathrm{log}} \hookrightarrow \mathcal{H}_{\mathrm{phys}}$ and with Obs the logical measurement invariant under the equivalence groupoid of physical variations. The four master formulas of the series that recur here are

$$\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x) \qquad \text{(stacks retain isotropy),} \tag{2}$$

$$\mathcal{H}_{\mathrm{phys}} \simeq (\mathcal{H}_{\mathrm{log}} \otimes \mathcal{H}_{\mathrm{gauge}}) \;\oplus\; \mathcal{H}_{\mathrm{err}} \qquad \text{(high-availability decomposition),} \tag{3}$$

$$\Delta(\mathcal{A}^{\mathfrak{m}}) = \textstyle\sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2} \qquad \text{(coaction / decomposition),} \tag{4}$$

$$\delta : \mathcal{L} \to \textstyle\bigwedge^2 \mathcal{L} \qquad \text{(cobracket).} \tag{5}$$

Equations (2)–(3) are proved in Sections 4 and 6; (4)–(5) are instantiated by the Connes–Kreimer coproduct in Section 8.

2.2 The three-level epistemic status system

The series' epistemic-honesty device, preserved verbatim, is a three-level status label attached to every dictionary entry.

Definition 2.2 (Status labels $\mathsf{S}/\mathsf{H}/\mathsf{P}$)**.** An entry of the representation dictionary carries a label:

- S *standard* use in mathematics or mathematical physics (e.g. TQFT as a functor $\mathrm{Bord}_n \to \mathbf{Vect}$; surface-code logical operators as $H_1(\Sigma; \mathbb{Z}_2)$);
- H a strong *heuristic* representation-dictionary entry (e.g. "gauge redundancy is fault-tolerant encoding" as a general slogan);
- P a *speculative* philosophical ontology (e.g. physical reality *as* motivic-categorical realization).

Composite labels S/H and H/P mark entries standard as pure mathematics but heuristic/speculative in their physical reading.

Remark 2.3 (Limitation: gauge redundancy is not physical symmetry)*.* We reproduce the series' governing limitation that is most relevant to Part VI: a *gauge redundancy* identifies different descriptions of the *same* physical content, whereas a *physical symmetry* maps one physical state to a *genuinely different* one. The mathematics of both is a group(oid) action;

the distinction is which action one declares to be "pure description." All status-S claims below concern the redundancy reading; the physical-symmetry reading is a separate modeling choice. This distinction is exactly what the *stack* (versus the coarse quotient) is built to keep track of.

3 Sheaves: compatible local data

3.1 Sites and presheaves

Definition 3.1 (Grothendieck topology, site)**.** A *Grothendieck topology* J on a small category $\mathcal{C}$ assigns to each object $U \in \mathcal{C}$ a collection $J(U)$ of *covering sieves* (equivalently, for the concrete presentations we use, covering families $\{U_i \to U\}_{i \in I}$), subject to:

1. (isomorphism) any isomorphism $\{V \xrightarrow{\sim} U\}$ covers U;
2. (stability) if $\{U_i \to U\}$ covers and $V \to U$ is any morphism, then the pullbacks $\{U_i \times_U V \to V\}$ cover V;
3. (transitivity) if $\{U_i \to U\}$ covers and each $\{U_{ij} \to U_i\}$ covers, then $\{U_{ij} \to U\}$ covers.

The pair $(\mathcal{C}, J)$ is a *site*. Strictly, the family-level axioms (isomorphism, stability, transitivity) above present a Grothendieck *pretopology*; the induced sieve-generated Grothendieck *topology* (maximality, stability, local character) yields the identical sheaf theory, and we use the two interchangeably, working throughout with the concrete covering families.

Example 3.2 (The topological site)**.** For a topological space Y, let $\mathrm{Op}(Y)$ be the poset of open sets with inclusions, and let $\{U_i \hookrightarrow U\}$ cover iff $\bigcup_i U_i = U$. This is the motivating site; "domains" U are regions, and a presheaf assigns data to each region.

Definition 3.3 (Presheaf)**.** A *presheaf* on $\mathcal{C}$ valued in **Set** is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. For $V \to U$ the induced map $F(U) \to F(V)$ is *restriction*, written $s \mapsto s|_V$. We write $\mathrm{PSh}(\mathcal{C})$ for the category of presheaves.

Physically (status S), a presheaf is a *local observable assignment*: to each region of spacetime, or each patch of a code's interaction hypergraph, it assigns the set of admissible local configurations, with restriction the operation of forgetting to a smaller region.

3.2 The sheaf condition

Definition 3.4 (Sheaf)**.** A presheaf F on a site $(\mathcal{C}, J)$ is a *sheaf* if for every covering family $\{U_i \to U\} \in J(U)$ the diagram

$$F(U) \xrightarrow{\ e\ } \prod_i F(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j) \tag{6}$$

is an equalizer, where $e(s) = (s|_{U_i})_i$, $p(s_i)_{ij} = (s_i|_{U_i \times_U U_j})$ and $q(s_i)_{ij} = (s_j|_{U_i \times_U U_j})$. Concretely: (*separated*) two global sections agreeing on a cover are equal; (*gluing*) any family of local sections $s_i \in F(U_i)$ that is *compatible* ($s_i|_{U_{ij}} = s_j|_{U_{ij}}$) descends to a unique global $s \in F(U)$ with $s|_{U_i} = s_i$.

Definition 3.5 (Stalk, obstruction)**.** On the topological site, the *stalk* of F at a point y is the filtered colimit $F_y = \varinjlim_{U \ni y} F(U)$; it is the germ of local data at the event y (status S). For a sheaf the equalizer (6) guarantees that compatible local sections always glue in degree 0; genuine *obstructions* live in higher sheaf cohomology $H^{n \geq 1}(U, F)$ — the failure of a globally defined section to lift along a surjection of sheaves, or of an abelian-group-valued Čech 1-cocycle to be a coboundary. Such a nonvanishing class represents an anomaly or topological obstruction (status S/H).

Proposition 3.6 (Sheafification)**.** *The inclusion* $\mathrm{Sh}(\mathcal{C}, J) \hookrightarrow \mathrm{PSh}(\mathcal{C})$ *of sheaves into presheaves admits a left adjoint* $(-)^{++}$*, the* sheafification*, and the unit* $\eta_F : F \to F^{++}$ *is universal among maps from* F *to sheaves. Sheafification is obtained by iterating the* plus construction $F^+(U) = \varinjlim_{R \in J(U)} \mathrm{Hom}_{\mathrm{PSh}}(R, F)$ *twice.*

Proof sketch. The plus construction F^+ is separated for any F and is a sheaf when F is already separated; hence F^{++} is a sheaf. For any sheaf G and map $f : F \to G$, the matching-family description of F^+ shows f factors uniquely through η, giving the adjunction $\mathrm{Hom}_{\mathrm{Sh}}(F^{++}, G) \cong \mathrm{Hom}_{\mathrm{PSh}}(F, G)$. This is the standard Grothendieck plus-construction; see Kashiwara–Schapira [1] and Vistoli [3]. We upgrade it to groupoid coefficients in Theorem 5.1. $\square$

The physical content of Theorem 3.6: *descent*, reconstructing a coordinate-independent global field from compatible local measurements, is the universal solution to the gluing problem. This is the set-valued shadow of the stack-valued statement we need for gauge fields, to which we now turn.

4 Stacks and gauge redundancy

A sheaf glues compatible local *data*. But a gauge field's local descriptions are related not by *equality* on overlaps but by *gauge transformations*: on U_{ij} the two restrictions differ by an element of the gauge group, and these transition data must satisfy a cocycle condition on triple overlaps. Recording this requires upgrading from set-valued to groupoid-valued data: a *stack*.

4.1 Fibered categories and descent

Definition 4.1 (Category fibered in groupoids)**.** A functor $\pi : \mathcal{F} \to \mathcal{C}$ is a *category fibered in groupoids* (CFG) if (i) for every morphism $f : V \to U$ in $\mathcal{C}$ and object ξ over U there is a *cartesian* arrow $f^*\xi \to \xi$ over f, and (ii) every arrow of $\mathcal{F}$ is cartesian. Then each fiber $\mathcal{F}(U) = \pi^{-1}(U)$ is a groupoid, and each $f : V \to U$ induces a pullback functor $f^* : \mathcal{F}(U) \to \mathcal{F}(V)$, functorial up to coherent natural isomorphism. We write $\mathcal{F}(U)$ for the groupoid of objects over U.

Physically, $\mathcal{F}(U)$ is the groupoid of local field configurations over the region U, with morphisms the gauge transformations between them (status S).

Definition 4.2 (Descent datum, stack)**.** Let $(\mathcal{C}, J)$ be a site and $\pi : \mathcal{F} \to \mathcal{C}$ a CFG. For a cover $\{U_i \to U\}$, a *descent datum* is a family $(\xi_i)_i$ with $\xi_i \in \mathcal{F}(U_i)$ together with isomorphisms $\varphi_{ij} : \xi_j|_{U_{ij}} \xrightarrow{\sim} \xi_i|_{U_{ij}}$ satisfying the *cocycle condition* $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ on triple overlaps U_{ijk}. Descent data and their morphisms form a groupoid $\mathrm{Desc}(\{U_i \to U\})$. The CFG $\mathcal{F}$ is a *stack* if for every cover the natural comparison functor

$$\mathcal{F}(U) \xrightarrow{\simeq} \mathrm{Desc}(\{U_i \to U\}) \tag{7}$$

is an *equivalence of groupoids* (every descent datum is effective, and this essentially uniquely). A CFG that satisfies (7) only as a fully faithful embedding (compatible local objects that glue are unique up to unique iso, but may fail to exist) is a *prestack*.

The comparison (7) is precisely the groupoid-valued upgrade of the sheaf equalizer (6): where the sheaf demanded a *unique* glued section, the stack demands a glued object *unique up to unique isomorphism*, and in addition demands that the isomorphisms themselves glue.

$$\begin{tikzcd}
\text{presheaf } F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set} \arrow[r, "\text{coefficients}"] \arrow[d, "\text{descent}"'] & \text{CFG } \pi : \mathcal{F} \to \mathcal{C} \arrow[d, "\text{descent}"] \\
\text{sheaf (unique gluing)} \arrow[r, "\text{coefficients}"'] & \text{stack (gluing up to unique iso)}
\end{tikzcd}$$

4.2 The quotient stack and isotropy

Definition 4.3 (Action groupoid and quotient stack)**.** Let a group G act on a space X. The *action groupoid* $X/\!\!/G$ has objects the points of X and morphisms $x \xrightarrow{g} g \cdot x$ for $g \in G$, with composition given by group multiplication. The *quotient stack* $[X/G]$ is the stackification of the prestack $U \mapsto \{G\text{-torsors } P \to U \text{ with } G\text{-map } P \to X\}$; over a point it is the groupoid $X/\!\!/G$. The *coarse quotient* X/G is the set of orbits $\pi_0(X/\!\!/G)$.

The two quotients are related by the coarse-space map $c : [X/G] \to X/G$, which is initial among maps from $[X/G]$ to sets/algebraic spaces. It is a bijection on isomorphism classes of points but collapses all automorphisms.

Proposition 4.4 (Stacks retain gauge information — T1, qualitative)**.** *If a physical configuration space is represented by descriptions X modulo gauge transformations G, and if stabilizer groups influence observables (anomalies, quantization, path-integral weights), then $[X/G]$ is a strictly more faithful representation object than X/G.*

Proof. The coarse quotient records orbits but forgets automorphism groups of representatives: $c(x) = c(x')$ iff $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(x')$, and c carries no further data. The quotient stack records objects, isomorphisms (gauge transformations), and self-isomorphisms (residual/isotropy symmetry). By Theorem 4.5 below, the self-isomorphisms at x are exactly $\mathrm{Stab}_G(x)$. Any observable that is a function of the isotropy group — e.g. an orbifold Euler characteristic $\chi_{\mathrm{orb}} = \sum_{[x]} 1/|\,\mathrm{Stab}_G(x)|$ or a path-integral measure weighting each orbit by $1/|\,\mathrm{Stab}_G(x)|$ — is therefore computable from $[X/G]$ but not from X/G, which has discarded the $|\,\mathrm{Stab}_G(x)|$ factors. Hence $[X/G] \to X/G$ is a genuine loss of information whenever some $\mathrm{Stab}_G(x)$ is nontrivial and enters an observable. $\square$

Theorem 4.5 (Isotropy of the quotient stack — T1, quantitative)**.** *For a G-action on X and any point $x \in X$, the automorphism group of x regarded as an object of $[X/G]$ is naturally isomorphic to the stabilizer:*

$$\mathrm{Aut}_{[X/G]}(x) \;\cong\; \mathrm{Stab}_G(x). \tag{8}$$

Moreover the residual gerbe *of $[X/G]$ at $[x]$ — the restriction of the stack to the orbit of x — is the classifying stack $\mathrm{B}\,\mathrm{Stab}_G(x) = [\mathrm{pt}/\,\mathrm{Stab}_G(x)]$. (The full étale-local model of $[X/G]$ near $[x]$ is the slice quotient $[S_x/\,\mathrm{Stab}_G(x)]$ for a $\mathrm{Stab}_G(x)$-invariant normal slice S_x to the orbit, reducing to $\mathrm{B}\,\mathrm{Stab}_G(x)$ exactly when the orbit is open.)*

Proof. Work in the action groupoid $X/\!\!/G$, which computes the fiber of $[X/G]$ over a point (Theorem 4.3). By definition a morphism $x \to x'$ in $X/\!\!/G$ is an element $g \in G$ with $g \cdot x = x'$, and composition is group multiplication. An *automorphism* of x is a morphism $x \to x$, i.e. an element $g \in G$ with $g \cdot x = x$; this is exactly the condition $g \in \mathrm{Stab}_G(x)$. The bijection $\mathrm{Aut}_{X/\!\!/G}(x) = \{g \in G : g \cdot x = x\} = \mathrm{Stab}_G(x)$ is a group isomorphism because composition of automorphisms is group multiplication, matching the group law on $\mathrm{Stab}_G(x)$. This proves (8).

For the residual gerbe: restrict the action groupoid to the single orbit of x. The orbit $\mathrm{Orb}_G(x) \cong G/\,\mathrm{Stab}_G(x)$ contributes no automorphisms (it is a free direction), while the residual symmetry is generated by $\mathrm{Stab}_G(x)$. The restricted groupoid is the transitive groupoid on $G/\,\mathrm{Stab}_G(x)$, which is equivalent as a groupoid to the one-object groupoid with automorphism group $\mathrm{Stab}_G(x)$, i.e. to $\mathrm{B}\,\mathrm{Stab}_G(x)$. Hence the restriction of $[X/G]$ to the orbit — its residual gerbe at $[x]$ — is $\mathrm{B}\,\mathrm{Stab}_G(x)$; the full étale-local model additionally records the normal slice as $[S_x/\,\mathrm{Stab}_G(x)]$. □

Corollary 4.6 (Faithfulness gap is exactly the isotropy)**.** *The fibers of $c : [X/G] \to X/G$ over $[x]$ are the classifying stacks $\mathrm{B}\,\mathrm{Stab}_G(x)$; equivalently, the information lost by passing from the stack to the coarse quotient is precisely the groupoid cohomology $H^\bullet(\mathrm{B}\,\mathrm{Stab}_G(x)) = H^\bullet_{\mathrm{grp}}(\mathrm{Stab}_G(x))$ of the isotropy groups. In particular X/G is faithful iff the action is free.*

Proof. Immediate from Theorem 4.5: over the orbit $[x]$ the coarse map forgets exactly the one-object groupoid $\mathrm{B}\,\mathrm{Stab}_G(x)$, whose invariants are the group cohomology of $\mathrm{Stab}_G(x)$. Freeness means all $\mathrm{Stab}_G(x)$ are trivial, so c is an equivalence. □

Example 4.7 (Electromagnetism as gauge redundancy)**.** Let X be the space of $U(1)$ connections (gauge potentials A) on a manifold and G the group of gauge transformations acting by $A \mapsto A + d\lambda$. The physical content is the curvature $F = dA$, and $d^2 = 0$ gives F gauge invariance. The coarse quotient X/G records the physical field strengths. The stack $[X/G]$ in addition records the automorphisms: for a generic connection the stabilizer is the constant gauge transformations $U(1)$, and this residual $U(1)$ isotropy is exactly the global symmetry that survives quantization and contributes the $1/\mathrm{vol}(G)$ Faddeev–Popov factors to the path integral. This is the canonical status-S illustration of Theorem 4.4.

Remark 4.8 (Classifying stack $\mathrm{B}G$). The special case $X = \mathrm{pt}$ gives $[\mathrm{pt}/G] = \mathrm{B}G$, the classifying stack, whose groupoid of points over any base is the groupoid of G-torsors. It classifies G-bundles (gauge fields), so the dictionary entry "classifying stack $\mathrm{B}G \leftrightarrow$ universal G-bundle object / gauge field classifier" (status S) is literally the statement that $\mathrm{Hom}(-, \mathrm{B}G)$ is the groupoid of gauge fields.

5 Stackification and the closure of the ladder

We now discharge Part I's forward reference. Part I posited a *representation prestack* $\mathsf{Rep}_{\rm phys} : \mathsf{Dom}^{\rm op} \to \mathbf{Grpd}$ assigning to each mathematical domain the groupoid of its physical representations, and asserted, by analogy, that this should be a *stack* (local representations glue up to gauge equivalence). Part I did not prove this. Here we do, via stackification.

Theorem 5.1 (Stackification of the representation prestack — T2)**.** *Let* (Dom, J) *be a site and* $\mathsf{Rep}_{\rm phys} : \mathsf{Dom}^{\rm op} \to \mathbf{Grpd}$ *the representation prestack of Part I (a CFG over* Dom*). Then:*

1. *There exists a stack* $\mathsf{Rep}^J_{\rm phys}$ *on* (Dom, J) *and a morphism of CFGs* $\eta : \mathsf{Rep}_{\rm phys} \to \mathsf{Rep}^J_{\rm phys}$ *such that for every stack* $\mathcal{G}$ *on* (Dom, J)*, precomposition*

$$\eta^* : \ \underline{\rm Hom}_{\rm Stack}(\mathsf{Rep}^J_{\rm phys}, \mathcal{G}) \ \xrightarrow{\ \simeq\ } \ \underline{\rm Hom}_{\rm CFG}(\mathsf{Rep}_{\rm phys}, \mathcal{G})$$

   is an equivalence of groupoids. That is, $\mathsf{Rep}^J_{\rm phys}$ *is the universal stack under* $\mathsf{Rep}_{\rm phys}$ *(*stackification*).*

2. $\mathsf{Rep}_{\rm phys}$ *is already a stack iff* η *is an equivalence.*

3. *Consequently, Part I's "representation stack" exists and is well-defined: it is* $\mathsf{Rep}^J_{\rm phys}$*. When the local representation data of Parts II–V (Betti/de Rham/Hodge realizations, cohomology, TQFT functors, HoTT identity data) are compatible on overlaps, they assemble into a global object of* $\mathsf{Rep}^J_{\rm phys}$*, exactly satisfying Part I's informal definition "entries glue not by equality but by gauge equivalence."*

Proof. We adapt the plus construction of Theorem 3.6 from set-valued to groupoid-valued (equivalently, $(2,1)$-truncated) coefficients. Define, for $U \in \mathsf{Dom}$,

$$\mathsf{Rep}^+_{\rm phys}(U) \ := \ \varinjlim_{\{U_i \to U\} \in J(U)} \ \mathrm{Desc}\Big(\{U_i \to U\}, \mathsf{Rep}_{\rm phys}\Big),$$

the (pseudo-)colimit over covers of the descent groupoids of Theorem 4.2, with transition functors the refinement functors of covers. This colimit is filtered: any two covers of U admit a common refinement (their pairwise pullbacks $\{U_i \times_U V_j \to U\}$), so the indexing category of covers is directed and the (pseudo-)colimit is well-behaved. Two applications, $\mathsf{Rep}^J_{\rm phys} := \mathsf{Rep}^{++}_{\rm phys}$, produce a stack: *(a)* $\mathsf{Rep}^+_{\rm phys}$ is a prestack (fully faithful descent) for any CFG $\mathsf{Rep}_{\rm phys}$, because morphisms in a descent groupoid are already determined locally and the cocycle condition makes them glue; *(b)* if $\mathsf{Rep}_{\rm phys}$ is a prestack then $\mathsf{Rep}^+_{\rm phys}$ is a stack, because effectivity of descent is achieved after one further plus step, exactly as in the 1-categorical case but now tracking the coherence isomorphisms φ_{ij} and their cocycle condition. Universality: given a stack $\mathcal{G}$ and a map $f : \mathsf{Rep}_{\rm phys} \to \mathcal{G}$, descent in $\mathcal{G}$ (Theorem 4.2) means every descent datum in $\mathcal{G}$ is effective, so f extends over each plus step uniquely up to unique coherent isomorphism; this yields the essentially unique factorization through η, i.e. the claimed equivalence of Hom-groupoids. Part (2) is immediate: if η is an equivalence then $\mathsf{Rep}_{\rm phys} \simeq \mathsf{Rep}^J_{\rm phys}$ is a stack, and conversely a stack is J-local so the plus construction is idempotent on it. Part (3) is the application: the compatibility of the local realization data supplied by Parts II–V is exactly a descent datum for $\mathsf{Rep}_{\text{phys}}$, hence an object of $\mathsf{Rep}^{J}_{\text{phys}}(U)$; this is the content of Part I's informal gluing axiom, now a theorem. This is the general stackification theorem for fibered categories over a site (Giraud [2]; Vistoli [3]), specialized to $\mathsf{Rep}_{\text{phys}}$. $\square$

$$\begin{tikzcd} \mathsf{Rep}_{\text{phys}} \arrow[r, "\eta"] \arrow[dr, "\forall f"'] & \mathsf{Rep}^{J}_{\text{phys}} \arrow[d, dashed, "\exists! \bar{f}\ \text{(up to iso)}"] \\ & \mathcal{G}\ \text{(any stack)} \end{tikzcd}$$

Corollary 5.2 (Closure of the ladder)**.** *The six modules I–VI are not merely an expository decomposition: Theorem 5.1 shows that Part I's opening definition (the representation stack) is a theorem provable from Part VI's descent machinery applied to the concrete data of Parts II–V. The ladder I → ⋯ → VI therefore closes back onto Part I. We nominate this as the headline "closure theorem" for the synthesis paper (Part VII).*

Remark 5.3 (Status). Theorem 5.1(1)–(2) is status S (the stackification theorem is standard; Giraud [2], Vistoli [3]). Part (3), the identification of the specific descent datum with Part I's "representation stack," is status $\mathsf{S/H}$: the mathematics is standard but the claim that the physical realization data of Parts II–V *is* a descent datum is the series' modeling hypothesis, and we flag it as such.

6 Quantum high-availability encodings

We now cross from geometry to quantum information. The same slogan (many physical representatives, one logical content, plus explicit redundancy data) becomes the theory of quantum error-correcting codes. We make the analogy precise and then, in Section 7, exhibit a case where it is a literal (status S) identity.

6.1 Codes as encodings with an equivalence groupoid

Definition 6.1 (Quantum high-availability representation)**.** A *quantum high-availability (QHA) representation* is an isometric encoding $\iota : \mathcal{H}_{\text{log}} \hookrightarrow \mathcal{H}_{\text{phys}}$ together with a groupoid $\mathcal{E}$ of *physical variations* (channels acting on $\mathcal{H}_{\text{phys}}$) such that:

1. every logical observable $\overline{O}$ (an operator on $\iota(\mathcal{H}_{\text{log}})$) is *invariant* under $\mathcal{E}$ in the sense that its induced action on the code space is unchanged by any variation in $\mathcal{E}$; and

2. detectable/correctable errors move the system within controlled equivalence or syndrome sectors, from which a recovery channel returns it to $\iota(\mathcal{H}_{\text{log}})$.

The redundancy $\dim \mathcal{H}_{\text{phys}} / \dim \mathcal{H}_{\text{log}}$ is the "availability": logical information survives loss of any physical subsystem in the correctable set.

This is the realization pipeline (1) with Real $= \iota$ and Obs the logical measurement; $\mathcal{E}$ plays the role of the gauge groupoid of Section 4. The subsystem structure is the high-availability decomposition (3):

$$\mathcal{H}_{\text{phys}} \;\simeq\; (\mathcal{H}_{\text{log}} \otimes \mathcal{H}_{\text{gauge}}) \;\oplus\; \mathcal{H}_{\text{err}}, \tag{9}$$

where $\mathcal{H}_{\text{gauge}}$ carries *gauge qubits* (redundant degrees of freedom that may be changed freely without affecting logical content) and $\mathcal{H}_{\text{err}}$ is the error sector detected by syndrome measurement. We stress the distinction between the two ways redundancy is realized. In a genuine *subsystem* (Bacon–Shor–type) or operator-algebra code the gauge factor is nontrivial, $\dim \mathcal{H}_{\text{gauge}} > 1$, and gauge operators act nontrivially within $\mathcal{H}_{\text{log}} \otimes \mathcal{H}_{\text{gauge}}$. In a *pure stabilizer* code the gauge factor degenerates, $\dim \mathcal{H}_{\text{gauge}} = 1$: the stabilizers act as the identity on the code space and the redundancy is carried entirely by the orthogonal syndrome sectors $\mathcal{H}_{\text{err}}$, i.e. by the many ways the full kinematic space projects onto $\iota(\mathcal{H}_{\text{log}})$. The parallel with $[X/G]$ is then this: $\mathcal{H}_{\text{log}}$ = logical/observable content (analogue of X/G), the redundancy the stack remembers (Stab/isotropy) appears as $\mathcal{H}_{\text{gauge}}$ for subsystem codes and as the $\mathcal{H}_{\text{err}}$ syndrome grading for pure stabilizer codes, and in both cases the encoding ι is the faithful (stack-level) representative that the coarse logical content alone forgets.

6.2 Stabilizer and operator-algebra codes

Definition 6.2 (Stabilizer code)**.** Let $\mathcal{P}_n$ be the n-qubit Pauli group. A *stabilizer code* [6] is specified by an abelian subgroup $S \subset \mathcal{P}_n$ containing no nontrivial phases, $S \cap \{\pm I, \pm iI\} = \{I\}$ (in particular $-I \notin S$, so the code space is nonzero); the *code space* is the simultaneous $+1$ eigenspace

$$\mathcal{H}_{\text{log}} \;=\; \{\, |\psi\rangle \in (\mathbb{C}^2)^{\otimes n} \;:\; s|\psi\rangle = |\psi\rangle \;\; \forall s \in S \,\}. \tag{10}$$

If S has $n-k$ independent generators then $\dim \mathcal{H}_{\text{log}} = 2^k$ (it encodes k logical qubits). The *logical operators* are elements of the normalizer $N(S)/S \subset \mathcal{P}_n/S$: Pauli operators commuting with all of S but not in S.

Lemma 6.3 (Knill–Laflamme conditions)**.** *Let P be the projector onto a code space $\mathcal{H}_{\text{log}} \subset \mathcal{H}_{\text{phys}}$ and $\{E_a\}$ a set of error operators. Then $\{E_a\}$ is correctable iff*

$$P E_a^\dagger E_b P \;=\; c_{ab}\, P \tag{11}$$

for a Hermitian matrix (c_{ab}). In words: the environment learns nothing about the logical state from the errors (no logical information leaks into which-error data), so a recovery channel exists.

Proof sketch. Necessity: if a recovery channel $\mathcal{R}$ satisfies $\mathcal{R}(E_a \rho E_b^\dagger) \propto \rho$ for all code states $\rho = P\rho P$, tracing against P forces (11). Sufficiency: diagonalize $(c_{ab}) = u\, d\, u^\dagger$; the combinations $F_\mu = \sum_a u_{a\mu} E_a$ satisfy $P F_\mu^\dagger F_\nu P = d_\mu \delta_{\mu\nu} P$, so the F_μ map $\mathcal{H}_{\text{log}}$ to mutually orthogonal subspaces isometrically (up to normalization); measuring which subspace (the *syndrome*) and inverting the corresponding isometry recovers the logical state. This is the standard Knill–Laflamme theorem [9]. □

Proposition 6.4 (Stabilizer syndromes form a sheaf; correction is the descent — T3, part i)**.** *For a stabilizer code with interaction hypergraph Γ (vertices = qubits, hyperedges = stabilizer* *supports, since a weight-w generator such as a surface-code star or plaquette touches $w > 2$ qubits; equivalently the bipartite Tanner graph with qubit- and check-nodes), the assignment*

$$U \longmapsto \{\textit{syndrome outcomes } \sigma : S_U \to \{\pm 1\} \textit{ consistent with } S\}, \qquad U \subseteq \Gamma,$$

where S_U are the stabilizer generators supported in U, is a sheaf on the poset of sub-hypergraphs, with restriction given by forgetting outcomes outside U. The substantive error-correction content is a second *descent, one level up: the map*

$$(\textit{global syndrome } \sigma : S \to \{\pm 1\}) \longmapsto (\textit{correctable global error class in } \mathcal{P}_n/S),$$

is well-defined precisely when the Knill–Laflamme conditions (Theorem 6.3) hold on the correctable set.

Proof. Restriction of syndrome outcomes to a sub-hypergraph is functorial and strictly compatible with further restriction, so the assignment is a presheaf (Theorem 3.3); moreover a family of local syndrome functions that agree on overlaps glues uniquely to a global syndrome function, since a syndrome is just an assignment of ± 1 to each generator — so syndromes trivially form a *sheaf*. This gluing is automatic and carries no error-correction content by itself. The non-trivial statement is the well-definedness of the recovery map: a globally consistent syndrome σ determines a *unique* correctable error class in $\mathcal{P}_n/S$ (equivalently a unique recovery up to a stabilizer) exactly when the Knill–Laflamme matrix (11) is nondegenerate on the correctable set. Thus the sheaf of syndromes is the easy layer, and correctability is the genuine descent from global syndrome data to a unique global logical correction. □

Remark 6.5 (Operator-algebra generalization). The subsystem decomposition (9) is the framework of *operator-algebra quantum error correction* (OAQEC): the correctable code is a von Neumann subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H}_{\text{phys}})$, and logical operators are $\mathcal{A}$, gauge operators its commutant restricted to the code, errors the complement. When $\mathcal{A}$ has nontrivial center one recovers gauge theories with a center (edge modes); this is the setting of Section 9.

7 Surface codes: gauge redundancy is literally homology

The analogy of Section 6 becomes a theorem for topological codes. Here the logical content is a *homology group* of the code surface: Part IV's dictionary entry "homology $H_n(X) \leftrightarrow$ conserved extended structures" realized literally, not analogically. This is the strongest status-S bridge in the series.

Definition 7.1 (Surface code)**.** The surface/toric code [4, 5] is defined as follows. Let Σ be a closed *connected* orientable surface of genus g with a cellulation (a lattice embedded in Σ). Place a qubit on each edge. For each vertex v define the *star operator* $A_v = \prod_{e \ni v} X_e$ (product over edges incident to v); for each face f define the *plaquette operator* $B_f = \prod_{e \in \partial f} Z_e$ (product over edges in the boundary of f). The stabilizer group is $S = \langle A_v, B_f \rangle$. These all commute (each A_v and B_f share an even number of edges), so (10) defines the code space.

Theorem 7.2 (Surface-code logical operators are the (co)homology $H_1, H^1(\Sigma;\mathbb{Z}_2)$ — T3)**.** *For the surface code on a genus-g surface Σ:*

1. *the code space has dimension 2^{2g}, i.e. $\mathcal{H}_{\log} \cong (\mathbb{C}^2)^{\otimes 2g}$ and the code encodes $k = 2g$ logical qubits;*

2. *the Z-type and X-type logical operators (modulo stabilizers) are respectively the nontrivial classes of $H_1(\Sigma;\mathbb{Z}_2)$ and the Poincaré-dual classes of $H^1(\Sigma;\mathbb{Z}_2)$, each isomorphic to $\mathbb{Z}_2^{2g}$, so that the full logical Pauli group modulo stabilizers is the $4g$-dimensional symplectic $\mathbb{Z}_2$-space*

$$N(S)/S \;\cong\; H_1(\Sigma;\mathbb{Z}_2) \;\oplus\; H^1(\Sigma;\mathbb{Z}_2) \;\cong\; \mathbb{Z}_2^{4g};$$

   each homological summand contributes the $2g$ generators of one Pauli type for the $2g$ logical qubits, and the intersection pairing supplies the symplectic (anti)commutation form;

3. *error strings that are boundaries $c = \partial b$ act trivially on the code space (they are "pure gauge"); only homologically nontrivial (noncontractible) strings implement logical operations.*

Proof. (1) Dimension count. There are $|E|$ qubits, and $|V|+|F|$ stabilizer generators A_v, B_f. They are not all independent: $\prod_v A_v = I$ and $\prod_f B_f = I$ (each edge appears in exactly two stars and two plaquettes over $\mathbb{Z}_2$), and these are the *only* relations — this holds precisely because Σ is a *closed* surface (no boundary), so every edge is shared by exactly two faces and two vertices. Hence the number of independent generators is $(|V|-1)+(|F|-1) = |V|+|F|-2$. The number of encoded qubits is

$$k \;=\; |E| - (|V|+|F|-2) \;=\; 2-(|V|-|E|+|F|) \;=\; 2-\chi(\Sigma) \;=\; 2-(2-2g) \;=\; 2g,$$

using the Euler characteristic $\chi(\Sigma) = |V|-|E|+|F| = 2-2g$. Thus $\dim \mathcal{H}_{\log} = 2^{2g}$.

(2) Homological identification. Work over $\mathbb{Z}_2$. A Z-type Pauli $\prod_{e\in c} Z_e$ is indexed by a 1-chain $c \in C_1(\Sigma;\mathbb{Z}_2)$. It commutes with every star A_v iff c is a *cycle* ($\partial c = 0$), since the number of edges of c at each vertex must be even. It lies in the stabilizer S (is a product of plaquettes) iff c is a *boundary* ($c = \partial b$ for a 2-chain b). Hence the Z-logical operators modulo stabilizers are

$$\ker \partial_1 / \operatorname{im} \partial_2 \;=\; H_1(\Sigma;\mathbb{Z}_2) \;\cong\; \mathbb{Z}_2^{2g}.$$

The same argument on the dual cellulation gives the X-logical operators as $H_1(\Sigma^\vee;\mathbb{Z}_2) \cong H^1(\Sigma;\mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$, Poincaré-dual to the Z-cycles. The full logical Pauli group modulo stabilizers is therefore the direct sum $N(S)/S \cong H_1 \oplus H^1 \cong \mathbb{Z}_2^{4g}$; the $\mathbb{Z}_2$-valued intersection pairing on $H_1 \times H^1$ supplies the nondegenerate symplectic form implementing the canonical (anti)commutation of the $2g$ logical qubit pairs (each qubit an X_i, Z_i pair drawn one from each homological summand).

(3) Boundaries are pure gauge. If $c = \partial b$ then $\prod_{e\in c} Z_e = \prod_{f\in b} B_f \in S$, so it acts as the identity on the code space (10). Thus a contractible error loop is undetectable and harmless, while only noncontractible loops — generators of H_1 — change the logical state. This is precisely the AT-library entry "boundary $c = \partial b \leftrightarrow$ pure gauge or physically null sector." ∎

$$
\begin{array}{ccccc}
C_2(\Sigma;\mathbb{Z}_2) & \xrightarrow{\partial_2} & C_1(\Sigma;\mathbb{Z}_2) & \xrightarrow{\partial_1} & C_0(\Sigma;\mathbb{Z}_2) \\
\Cup & & \Cup & & \Cup \\
\text{plaquettes } B_f & & Z\text{-strings} & & \text{stars } A_v
\end{array}
$$

Corollary 7.3 (Gauge redundancy $\leftrightarrow$ quantum high availability, literal case)**.** *The surface code realizes* (9) *and* (8) *simultaneously. The logical content is the homology $\mathcal{H}_{\log} \cong (\mathbb{C}^2)^{\otimes 2g}$ with logical operators graded by $H_1(\Sigma;\mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$. Because the surface code is a* pure *stabilizer code, the gauge factor is trivial (*$\dim \mathcal{H}_{\text{gauge}} = 1$*): the contractible-loop (boundary) operators lie* in *the stabilizer S and act as the identity on the code space, so they are the "pure gauge" null sector rather than operators on a nontrivial tensor factor. The genuine redundancy is therefore carried by the syndrome grading $\mathcal{H}_{\text{err}}$: the violated stars/plaquettes A_v, B_f label the orthogonal error sectors through which the full $2^{|E|}$-dimensional kinematic space projects onto the 2^{2g}-dimensional code space. (A Bacon–Shor–type subsystem code on Σ would instead populate a nontrivial $\mathcal{H}_{\text{gauge}}$; see the remark on OAQEC below.) Hence "gauge redundancy is to physics what fault-tolerant encoding is to quantum information" is,* for surface codes*, a theorem (status* S*), not a slogan: the gauge-null (boundary) sector and the logical sector are the two homological strata* $\operatorname{im}\partial_2$ *and* $H_1(\Sigma;\mathbb{Z}_2)$ *of one and the same chain complex.*

Remark 7.4 (Cross-module reuse)*.* Theorem 7.2 reuses Part IV's homology dictionary verbatim ($H_n(X) \leftrightarrow$ conserved extended structures) and instantiates Part I's Equivalence axiom (gauge-equivalent = homologous). It is the module's central case study and the reason Part VI has the most cross-references of any module.

8 Hopf-algebraic decomposition: renormalization as antipode

The second reused backbone is coalgebraic. Part II introduced the Goncharov [8] Lie coalgebra and the coaction (4) as the "anatomy" of amplitudes. Part VI uses the *same* formal operation (a graded connected Hopf-algebra coproduct) on a different combinatorial datatype (Feynman graphs / rooted trees), where the antipode computes renormalization counterterms.

Definition 8.1 (Hopf algebra of rooted trees)**.** Let H be the free commutative algebra over $\mathbb{Q}$ on rooted trees (forests as products), graded by the number of vertices (loop number). The *coproduct* is the sum over admissible cuts:

$$\Delta(\Gamma) \;=\; \Gamma\otimes 1 \;+\; 1\otimes\Gamma \;+\; \sum_{\text{cuts } c} P^c(\Gamma)\otimes R^c(\Gamma), \tag{12}$$

where R^c is the component containing the root and P^c the pruned forest. H is a graded connected bialgebra; connectedness ($H_0 = \mathbb{Q}$) forces a unique *antipode* S making H a Hopf algebra, given recursively by

$$\mathrm{S}(\Gamma) \;=\; -\Gamma \;-\; \sum_{\text{cuts } c} \mathrm{S}\Big(P^c(\Gamma)\Big)\, R^c(\Gamma). \tag{13}$$

Definition 8.2 (Feynman-rules character and Birkhoff decomposition)**.** A regularized *Feynman-rules character* is an algebra map $\varphi : H \to \mathbb{C}(\!(z)\!)$ (formal Laurent series in the dimensional-regularization parameter z), an element of the group $G = \mathrm{Hom}_{\mathrm{alg}}(H, \mathcal{A})$ of $\mathcal{A}$-valued characters under the convolution product $\varphi \star \psi = m_{\mathcal{A}} \circ (\varphi \otimes \psi) \circ \Delta$. The minimal-subtraction splitting $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ (pole part vs. holomorphic part) is a Rota–Baxter structure of weight -1.

Theorem 8.3 (Antipode = counterterm; renormalization = Birkhoff decomposition — T4)**.** *Let $\varphi : H \to \mathcal{A}$ be a Feynman-rules character valued in a commutative Rota–Baxter algebra $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ with projection R onto $\mathcal{A}_-$. Then φ admits a unique* Birkhoff decomposition $\varphi = \varphi_-^{\star -1} \star \varphi_+$ *in G, with the* counterterm φ_- *and* renormalized character φ_+ *given recursively by Bogoliubov's formulas*

$$\varphi_-(\Gamma) \;=\; -R\Big(\varphi(\Gamma) + \sum \varphi_-(P^c\Gamma)\,\varphi(R^c\Gamma)\Big), \tag{14}$$

$$\varphi_+(\Gamma) \;=\; (1 - R)\Big(\varphi(\Gamma) + \sum \varphi_-(P^c\Gamma)\,\varphi(R^c\Gamma)\Big). \tag{15}$$

The counterterm $\varphi_- = \mathrm{S}_R^{\varphi}$ is the twisted (Rota–Baxter) antipode — an operator on the character group $\mathrm{Hom}(H, \mathcal{A})$*, not a precomposition with a map on H, since R acts on the target algebra $\mathcal{A}$ — and φ_+ is the BPHZ-renormalized amplitude. Thus renormalization is exactly the Hopf-algebraic decomposition* (4) *applied to the divergence structure.*

Proof sketch. Since H is graded connected, the convolution group G has a well-defined exponential/ logarithm and every character has a $\star$-inverse computed by the antipode (13). Writing $\varphi_- = e \circ \epsilon - R \circ (\varphi_- \star (\varphi - e \circ \epsilon))$ and solving degree by degree gives (14); the Rota–Baxter identity [15]

$$R(a)R(b) + R(ab) = R\Big(R(a)b + aR(b)\Big)$$

is precisely what makes φ_- multiplicative (a character), so the decomposition lands in the group G. The holomorphic part $\varphi_+ = \varphi_- \star \varphi$ is then finite at $z = 0$ and equals the forest formula of BPHZ. Identifying the recursion with the antipode recursion (13) twisted by R gives $\varphi_- = \mathrm{S}_R^{\varphi}$, the twisted antipode on $\mathrm{Hom}(H, \mathcal{A})$. This is the Connes–Kreimer Riemann–Hilbert theorem [7]. □

Corollary 8.4 (Shared Hopf backbone of Parts II and VI)**.** *Both the Goncharov coaction (Part II) and the Connes–Kreimer coproduct (Part VI) are coproducts of graded connected Hopf algebras; they are the same formal operation* (4) *applied to two combinatorial datatypes (polylog symbols vs. rooted trees), with two physical readings (amplitude anatomy vs. renormalization subtraction). The synthesis paper should present these as one "Hopf algebra of decomposition" chapter.*

9 Holographic quantum error correction: strengthening the bridge

The slogan "gauge redundancy is to physics what fault-tolerant encoding is to quantum information" (Theorem 7.3 makes it a theorem for surface codes) is, in full generality, status

H. The most rigorous body of work strengthening it toward status S is *holographic quantum error correction*. We summarize it and label its epistemic status carefully.

9.1 Bulk locality as an error-correcting code

In the AdS/CFT correspondence a bulk (gravitational) operator is redundantly encoded on the boundary conformal field theory. Almheiri, Dong, and Harlow [10] observed that this redundancy is precisely that of a quantum error-correcting code: a bulk local operator in the interior can be reconstructed on any of several boundary subregions, exactly as a logical operator of a code has representatives on several physical subsystems [13]. Access to only a boundary subregion is like having part of the code erased; the operator can still be reconstructed provided the subregion contains the relevant entanglement wedge, a fact that can be derived from bulk modular flow [14].

Principle 9.1 (Holographic code dictionary, status S/H). Under the entanglement-wedge reconstruction of AdS/CFT:

bulk effective field theory	$\leftrightarrow$	logical Hilbert space $\mathcal{H}_{\log}$,
boundary CFT	$\leftrightarrow$	physical Hilbert space $\mathcal{H}_{\text{phys}}$,
bulk gauge/diffeomorphism redundancy	$\leftrightarrow$	code redundancy,
radial/holographic direction	$\leftrightarrow$	redundancy depth,
Ryu–Takayanagi entropy	$\leftrightarrow$	code subalgebra / entropy.

The Pastawski–Yoshida–Harlow–Preskill (HaPPY) tensor-network codes [11] are explicit toy models realizing this dictionary.

Remark 9.2 (Harlow's theorem: RT from QEC). Harlow [12] proved that a quantum-corrected Ryu–Takayanagi formula holds in *any* operator-algebra quantum error-correcting code (subalgebra codes, Theorem 6.5), and that when the boundary subalgebras have nontrivial center the code describes a *gauge theory with edge modes*. This is the sharpest known sense in which bulk gauge redundancy *is* the redundancy of a code: the center of the subalgebra is the gauge (superselection/edge-mode) data, exactly the $\mathcal{H}_{\text{gauge}}$ factor of (9). Status: S as a theorem about OAQEC codes; S/H as a claim about physical quantum gravity, where finite-N corrections may modify the picture.

9.2 Epistemic assessment

- For **surface/topological codes**, "gauge redundancy = QHA" is a theorem (Theorem 7.3): status S.
- For **OAQEC/subalgebra codes with center**, the identification "gauge (edge mode) data = subalgebra center" is a theorem (Harlow): status S.
- For **physical AdS/CFT quantum gravity**, the holographic-code picture is strongly supported but subject to finite-N/gauge-invariance caveats: status S/H.
- The fully general slogan across *all* gauge theories remains a guiding heuristic: status H.

We therefore report the bridge as a spectrum from theorem (surface codes) to heuristic (fully general), rather than as a single claim, exactly the discipline the series' status system was designed to enforce.

10 Results

We collect the principal results and their epistemic status.

Theorem 10.1 (Summary of Part VI)**.** *Let $(\mathcal{C}, J)$ be a site, G a group acting on X, and consider stabilizer/surface codes as above. Then:*

1. (T1, Theorem 4.5, status S) $\mathrm{Aut}_{[X/G]}(x) \cong \mathrm{Stab}_G(x)$*, and* $[X/G]$ *is strictly more faithful than* X/G *exactly on the isotropy* $H^{\bullet}_{\mathrm{grp}}(\mathrm{Stab}_G(x))$*.*

2. (T2, Theorem 5.1, status S; application S/H) *the representation prestack* $\mathsf{Rep}_{\mathrm{phys}}$ *admits a universal stackification* $\mathsf{Rep}^{J}_{\mathrm{phys}}$*, which is Part I's representation stack; the ladder closes (Theorem 5.2).*

3. (T3, Theorem 7.2, status S) *surface-code Z- and X-type logical operators are* $H_1(\Sigma; \mathbb{Z}_2)$ *and* $H^1(\Sigma; \mathbb{Z}_2)$*, each* $\cong \mathbb{Z}_2^{2g}$*, so* $N(S)/S \cong \mathbb{Z}_2^{4g}$ *encodes* $2g$ *logical qubits; boundaries are pure gauge. Hence gauge redundancy = QHA is a theorem for surface codes (Theorem 7.3).*

4. (T4, Theorem 8.3, status S) *the Connes–Kreimer antipode computes the BPHZ counterterm and renormalization is the Birkhoff decomposition of the Feynman-rules character; Parts II and VI share one Hopf-algebra backbone (Theorem 8.4).*

Proof. Each part is proved in the referenced statement: (1) Theorem 4.5, Theorem 4.6; (2) Theorem 5.1, Theorem 5.2; (3) Theorem 7.2, Theorem 7.3; (4) Theorem 8.3, Theorem 8.4. □

Mathematics	**Physical representation**	**QI interpretation**	**Status**
Presheaf	local observable assignment	local syndrome data	S
Sheaf	fields from local data	syndrome consistency (local gluing)	S
Descent	coordinate-independent physics	correctable global error / recovery	S
Stack $[X/G]$	gauge field w/ redundancy	encoding + equivalences	S
$\mathrm{Aut}_{[X/G]}(x) \cong \mathrm{Stab}_G(x)$	residual/isotropy symmetry	error-detecting redundancy	S
Classifying stack $\mathrm{B}G$	gauge-field classifier	code-family classifier	S
$H_1(\Sigma; \mathbb{Z}_2)$	conserved cycles	surface-code logical qubits	S
Boundary $c = \partial b$	pure gauge / null	trivial (in-stabilizer) error	S
Hopf coproduct Δ	decomposition	nested-error bookkeeping	S
Antipode S	renormalization inverse	counterterm	S/H
Gauge = QHA (general)	redundancy = fault tolerance	holographic code	H

Table 1: Part VI representation dictionary with status labels (excerpt of Appendix F, augmented with quantum-information column). Composite labels mark entries standard as mathematics but heuristic in their fully general physical reading.

11 Discussion

11.1 How Part VI composes on Parts I–V

Part VI is the module with the most cross-references, as befits a capstone:

- **From Part V** [20] it takes fibered categories, groupoids, and the $(2,1)$-categorical machinery needed for Theorem 4.2 and Theorem 5.1; univalence (Part V) is the type-theoretic shadow of the descent equivalence (7).
- **To Part I** [16] it delivers Theorem 5.1, discharging the forward reference and closing the ladder (Theorem 5.2).
- **From Part IV** [19] it reuses homology: Theorem 7.2 makes Part IV's dictionary entry literal.
- **From Part II** [17] it reuses the Hopf-algebraic coproduct: Theorem 8.3 is Part II's decomposition axiom applied to divergences (Theorem 8.4).
- **From Part III** [18] it reuses moduli stacks: $[X/G]$ is the gauge-theoretic special case of Part III's moduli stacks.

11.2 Limitations

We reproduce and address the series' limitations (Part I, §0.5):

1. *Not every analogy is a theory.* Status labels are part of the formalism: Table 1 carefully distinguishes the status-S surface-code identity (Theorem 7.3) from the status-H general slogan (Section 9).
2. *Gauge redundancy $\neq$ physical symmetry* (Theorem 2.3). The stack $[X/G]$ is the right object *only* when G is declared pure description; the same mathematics with G a physical symmetry would give a genuinely different physics. This modeling choice is not decided by the mathematics.
3. *No single universal functor* $\mathsf{Math} \to \mathsf{Phys}$. Theorem 5.1 is *local* (indexed by the site Dom); it does not assert one global functor. This respects the series' deliberate domain-indexed design.
4. *Finite-N caveats.* The holographic-code bridge (Section 9) is subject to gauge-invariance and finite-N corrections; we label it S/H accordingly and do not overclaim it as a theorem about physical quantum gravity.

11.3 Open problems

- Beyond surface codes and OAQEC, is there a general theorem making "gauge redundancy = fault-tolerant encoding" status S for an arbitrary gauge theory? The holographic literature (Harlow; entanglement-wedge reconstruction) is the most promising route.

- Can the stackification $\mathsf{Rep}^J_{\text{phys}}$ (Theorem 5.1) be carried out explicitly for a concrete site of physical theories (factorization homology; Schreiber's cohesive-topos program)?
- Is there a single graded connected Hopf algebra unifying the Goncharov and Connes–Kreimer coproducts (Theorem 8.4)?

12 Conclusion

Part VI has formalized sheaves, stacks, gauge redundancy, and quantum high-availability as one construction seen four ways: *many local representatives, one invariant logical content, plus explicit data recording the redundancy.* We proved that the quotient stack retains isotropy (T1), that the representation prestack stackifies to Part I's representation stack and thereby closes the modular ladder (T2), that surface-code logical operators are literally $H_1(\Sigma; \mathbb{Z}_2)$ so that "gauge redundancy = quantum high-availability" is a theorem for surface codes (T3), and that Connes–Kreimer renormalization is Hopf-algebraic decomposition sharing Part II's coalgebraic backbone (T4). We surveyed holographic quantum error correction as the rigorous frontier strengthening the general bridge, with each claim's epistemic status ($\mathsf{S}/\mathsf{H}/\mathsf{P}$) explicitly labeled.

The four closing theses of the series specialize here to:

> *symmetry redundancy is extra descriptive freedom that does not change physical information; a gauge field is a physical field represented by locally redundant descriptions; a stack is the mathematical object that remembers local descriptions, gluing, and redundancy symmetries; quantum high availability is logical information protected by encoding it across redundant physical degrees of freedom.*

Gauge redundancy is to physics what fault-tolerant encoding is to quantum information: proven for surface codes, conjectured in general, and now formalized as the capstone module of the representation library. The synthesis paper (Part VII) will assemble the closure theorem (Theorem 5.2) with the four master formulas and the status-label statistics into a single recomposition of the ladder.

Companion formalizations

The constructions above are encoded in companion code.

- **Haskell** (`src/sheaves-stacks-gauge-quantum-ha/`): modules `Sheaf`, `Stack` (coarse vs. stack quotient, returning the retained stabilizer), `QHA` (encoding, syndrome, recovery), `Stabilizer` (surface-code stabilizers and H_1 logicals), and `Hopf` (Connes–Kreimer coproduct and antipode). The `Main` module demonstrates T1 (isotropy retained), T3 (genus-g code dimension 2^{2g}), and T4, whose antipode passes the Hopf axiom $m(\mathrm{S} \otimes \mathrm{id})\Delta = 0$.
- **Lean** (`lean/sheaves-stacks-gauge-quantum-ha/`): a Mathlib-free, best-effort sketch of the types `SheafOnSite`, `StackOnSite`, `QuotientStack`, and

`StabilizerCode`, together with the theorem signatures `aut_equiv_stabilizer` (T1) and `surfaceCode_logical_eq_h1Rank` (T3); the file type-checks, with the substantive proofs recorded as deferred `sorry` goals in the idiomatic sketch style.

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