A Modular Representation Synthesis: Composing Six Mathematical Faculties into Physical Representation

Part VII (Synthesis) of *A Math→Physics Representation Library*

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Abstract

This capstone part recomposes the six modular papers of *A Math→Physics Representation Library* into a single coherent account of physical representation. We take the *representation stack* and the *realization pipeline* $\mathrm{Obs}_\alpha(M) = \mathrm{Obs}(\mathrm{Real}_\alpha(\Phi(M)))$ of Part I as the spine, and exhibit Parts II–VI as successive concrete instantiations and rigorous discharges of that spine. The organizing device is the *modular composition ladder* I → II → III → IV → V → VI: at each rung a new mathematical structure is introduced (motives and coactions; schemes and moduli; (co)homology and TQFT; categories, functors, and univalence; sheaves, stacks, and quantum codes), and a single *emergent representational property* appears only after composition: coalgebraic anatomy, geometric variation, functorial conservation, universal compositional grammar, and gauge/fault-tolerant high availability. We stress that this is a *modular* framework, not a monolithic unified theory: each module is defensible on its own, and emergence is a property *of the composition* that we track explicitly at each level.

Our central result is the **Closure Theorem**: the representation stack posited informally in Part I is a genuine mathematical object, provably closed under the operations of all six faculties, and recovered exactly as the descent-theoretic stackification of the representation prestack built from the concrete data of Parts II–VI. Its keystone is the isotropy correspondence $\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x)$, one theorem with four independent proofs (informal, algebro-geometric, type-theoretic, descent-theoretic), of which *gauge redundancy* ≅ *quantum high availability* is the physical face: many physical representatives, one logical/observable content. Throughout we carry the tripartite epistemic status labels S (standard), H (heuristic), P (speculative), and we close with a quantitative status-label census across the 268 dictionary entries of the library, a verbatim reproduction of the programme's five limitations, and a forward research agenda.

Keywords: representation stack, realization pipeline, motives and periods, positive geometry, topological quantum field theory, homotopy type theory, univalence, descent and stacks, gauge redundancy, quantum error correction, modular composition, emergence.

Contents

1 Introduction: a modular library, not a monolith

1.1 The programme

The six preceding parts of this series each formalize one mathematical faculty as a *physical-representation module*:

Part i. *Foundations: the Representation Stack and the Realization Pipeline.* The common substrate: a domain-indexed category Dom of mathematical domains, per-domain categories Math_d, a category Phys of physical representation targets, representation entries $E = (M, P, \tau, \sigma)$ carrying an epistemic status σ, the groupoid-valued pseudofunctor $\mathsf{Rep}_{\mathrm{phys}} : \mathsf{Dom}^{\mathrm{op}} \to \mathsf{Gpd}$ (the representation *prestack*), the four ontology axioms, and the master realization pipeline.

Part ii. *Motives, Periods, and Amplitudes: the coalgebraic anatomy of physical representation.* Motives $\mathrm{Mot}(X)$ as functional essence, period data $\Pi = (X, D, \omega, \Gamma)$, motivic amplitude objects $\mathcal{A}^{\mathsf{m}}$ with $\mathrm{per}(\mathcal{A}^{\mathsf{m}}) = \mathcal{A}$, the coaction Δ and cobracket δ, the Goncharov Lie coalgebra.

Part iii. *Algebraic Geometry: spaces of physical possibility.* Schemes, $\mathrm{Spec}\, R$, moduli spaces and stacks $[X/G]$, variations of Hodge structure, the Gauss–Manin connection, and positive geometries with recursive residues.

Part iv. *Algebraic Topology: conserved global information.* (Co)homology, characteristic classes, bordism, and topological quantum field theory (TQFT) as a symmetric monoidal functor $Z : \mathsf{Bord}_n \to \mathsf{Vect}$.

Part v. *Category Theory and Homotopy Type Theory: composition and identity.* Functors, natural transformations, monoidal/dagger-compact structure, identity types, and univalence.

Part vi. *Sheaves, Stacks, Gauge Redundancy, and Quantum High Availability.* Sheaves and stacks as (pseudo)functors satisfying descent, the quotient stack $[X/G]$, stabilizer, subsystem, and surface codes, and Hopf-algebraic renormalization.

The founding thesis of the library [1] is representational rather than metaphysical: many physical quantities are obtained as *realizations of structured mathematical information*, and the discipline lies in *tracking which translations are which*. This is why every module carries the tripartite status system: S for a standard mathematics/mathematical-physics correspondence, H for a strong but heuristic dictionary entry, and P for a speculative ontological extension. The labels are part of the formalism, not decoration.

1.2 Modular, not unified

We emphasize at the outset a point that governs the entire synthesis. This is a *modular* framework. The six modules are not chapters of a single monolithic "theory of everything mathematical" collapsing all of mathematics onto all of physics through one universal functor; indeed such a functor is almost certainly not well defined (Section 9, Limitation 3). Instead, each module is a self-contained, independently defensible representation faculty, and the object of this synthesis is to exhibit how the faculties *compose*, hierarchically and functorially, and to isolate, at each rung of the composition ladder, the *emergent representational property* that exists only in the composite and is absent from any single factor. Composition is what produces these emergent properties, and it does so without collapsing the six modules into a single primitive.

1.3 The spine and the ladder

The synthesis has two organizing structures, which we develop in tandem.

The **spine** is Part I's realization pipeline

$$\mathrm{Obs}_\alpha(M) \;=\; \mathrm{Obs}\big(\mathrm{Real}_\alpha(\Phi(M))\big), \qquad M \in \mathsf{Math}_d, \tag{1}$$

together with the prestack $\mathsf{Rep}_{\mathrm{phys}} : \mathsf{Dom}^{\mathrm{op}} \to \mathsf{Gpd}$. Every later module supplies a concrete instance of (1): Φ becomes "take the motive," Real_α becomes a Betti/de Rham/Hodge/étale realization, a Gauss–Manin transport, a TQFT functor, an internal type-theoretic construction, or a quantum encoding, and Obs becomes a period, an index, a dimension/trace, or a syndrome measurement.

The **ladder** is the hierarchical build order

$$\mathrm{I} \longrightarrow \mathrm{II} \longrightarrow \mathrm{III} \longrightarrow \mathrm{IV} \longrightarrow \mathrm{V} \longrightarrow \mathrm{VI},$$

in which each module strictly presupposes the previous ones and adds one new structure. The ladder is *circular*: Part I posits that $\mathsf{Rep}_{\mathrm{phys}}$ is "stack-like" but defers the definition of a stack to a forward reference; Part VI supplies exactly that definition (fibered category + descent) and proves the forward reference. Closing this loop is the business of the Closure Theorem (Section 6).

1.4 Contributions of this synthesis

1. We recompose the ladder rung by rung (Section 3), stating for each level the new structure, the Part I axiom it instantiates or discharges, and the single emergent property it contributes.

2. We assemble a *unified mathematical framework* (Section 4), a total representation category $\int \mathsf{Rep}_{\text{phys}}$ with a global realization functor, and a *master commuting diagram* (Figure 2) tying the six faculties to the single target Phys and back to Part I.

3. We isolate the cross-cutting themes (Section 5): one realization formula instantiated six times; one Hopf/coalgebra of decomposition subsuming coaction, cobracket, boundary operator, spectral sequence and antipode; a functoriality gradient of increasing concreteness; the reuse of homology as a literal error-correcting resource; and the isotropy theorem in four guises.

4. We state and rigorously assemble the **Closure Theorem** (Section 6), with the four-proof isotropy correspondence as its axis and *gauge redundancy* $\cong$ *quantum high availability* as its keystone.

5. We provide a quantitative *status-label census* (Section 8), a verbatim reproduction and per-module discussion of the programme's *limitations* (Section 9), and an *open-problem research agenda* (Section 10).

Cross-references of the form "(III/T2)" denote candidate Theorem 2 of Part III, etc.; all such statements are reproduced or summarized here so that the synthesis is self-contained.

2 The representation stack as spine

We recall the substrate of Part I in the exact form the later modules build upon. Nothing in this section is new; it fixes the notation and the axioms that Sections 3, 4 and 6 manipulate.

2.1 Objects

Dom is a category (in fact a site) of mathematical domains: algebraic geometry, algebraic topology, category theory, homotopy type theory, sheaf/stack theory, motivic period theory, and so on. For each domain $d \in \mathsf{Dom}$ there is a category (or higher category) Math_d of mathematical structures of that domain. Phys is a category (bicategory, ∞-category) of *physical representation targets*: state spaces, observable algebras, process categories, gauge moduli stacks, quantum code spaces, measurement outcomes. Realization channels are indexed by a label $\alpha \in \{\text{Betti}, \text{de Rham}, \text{Hodge}, \text{étale}, p\text{-adic}, \text{quantum}, \text{thermodynamic}, \dots\}$. The bracket $[\![M]\!]_{\text{phys}}$ denotes the physical representation assigned to M under a chosen interpretation.

2.2 The four definitions

Definition 2.1 (Representation entry)**.** A *representation entry* is a quadruple $E = (M, P, \tau, \sigma)$ with $M \in \mathsf{Math}_d$ a mathematical structure, $P \in \mathsf{Phys}$ a proposed physical representation, τ a semantic translation datum (a functor, natural transformation, equivalence class of models, interpretation map, or weaker relation), and $\sigma \in \{\mathsf{S}, \mathsf{H}, \mathsf{P}\}$ the epistemic status.

Definition 2.2 (Representation library)**.** A *representation library* is a collection of representation entries closed, when defined, under: (1) restriction to subdomains; (2) transport

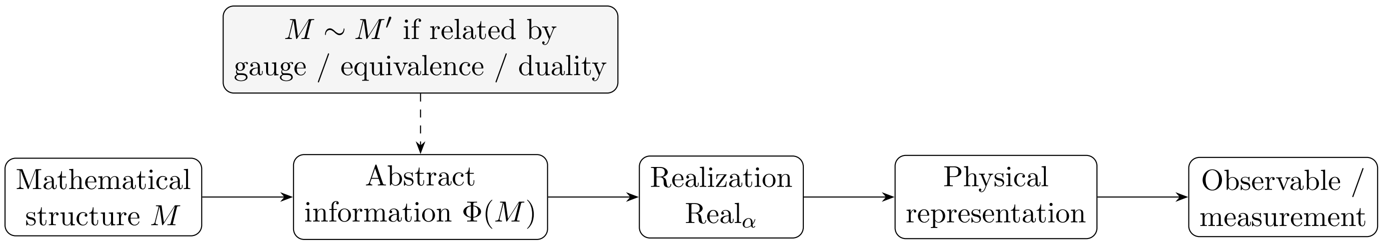

Figure 1: The spine: Part I's realization pipeline, computing $\mathrm{Obs}_\alpha(M) = \mathrm{Obs}(\mathrm{Real}_\alpha(\Phi(M)))$, with the gauge/equivalence datum (Axiom 2) feeding the abstract-information step. Every module of the library is a concrete instance of this diagram.

along equivalence of mathematical structures; (3) composition of compatible translations; (4) decomposition through boundary/coproduct/localization; (5) realization into observables.

Definition 2.3 (Representation prestack)**.** A pseudofunctor $\mathsf{Rep}_{\mathrm{phys}} : \mathsf{Dom}^{\mathrm{op}} \to \mathsf{Gpd}$ assigning to each domain d a groupoid of representation entries, and to each refinement $u : e \to d$ a restriction functor $u^* : \mathsf{Rep}_{\mathrm{phys}}(d) \to \mathsf{Rep}_{\mathrm{phys}}(e)$.

Definition 2.4 (Representation stack)**.** Given a Grothendieck topology on Dom, $\mathsf{Rep}_{\mathrm{phys}}$ is a *representation stack* if compatible families of local representation entries glue to global entries uniquely up to coherent isomorphism. Automorphisms represent gauge changes, dualities, code stabilizers, or other physically irrelevant equivalences.

2.3 The pipeline and the axioms

The universal formula of the whole programme is the *realization pipeline*

$$M \xrightarrow{\Phi} \text{abstract information } \Phi(M) \xrightarrow{\mathrm{Real}_\alpha} \text{physical representation} \xrightarrow{\mathrm{Obs}} \text{measurement data},$$
$$\mathrm{Obs}_\alpha(M) = \mathrm{Obs}(\mathrm{Real}_\alpha(\Phi(M))). \tag{2}$$

Axiom 1 (Realization)**.** observable $= \mathrm{Obs} \circ \mathrm{Real}_\alpha$(abstract structure).

Axiom 2 (Equivalence / gauge)**.** Mathematical descriptions related by an equivalence lying in the automorphism/gauge groupoid of a representation entry determine the same physical content at the chosen observational level.

Axiom 3 (Locality / descent)**.** Physical representations must be locally assignable and globally coherent; failure of descent is an obstruction, anomaly, or missing boundary datum.

Axiom 4 (Decomposition)**.** Compositional or period-like observables admit a decomposition operation (boundary, coproduct, coaction, spectral sequence, factorization) revealing lower-level informational pieces.

The master (spine) diagram of Part I, which every module instantiates, is reproduced in Figure 1.

2.4 The status calculus

The status labels are ordered $\mathsf{S} \preceq \mathsf{H} \preceq \mathsf{P}$ ("standard is stronger than heuristic is stronger than speculative"). Part I (I/T3) makes this a *calculus*: on $\{\mathsf{S}, \mathsf{H}, \mathsf{P}\}$ put the commutative idempotent monoid structure with unit S and product max under $\preceq$. Then the composite of representation entries E_1, E_2 (when $\tau_2 \circ \tau_1$ is defined) has status $\sigma_1 \vee \sigma_2 := \max(\sigma_1, \sigma_2)$: *status cannot improve under composition; a chain of translations is only as reliable as its weakest link*. This observation, which is checkable and lends itself to mechanized propagation, governs every composite we form in Sections 3 and 6, and it is the reason the Closure Theorem must carefully track the status of each ingredient.

3 The modular composition ladder

We now traverse the ladder. For each rung we state (i) the new structure introduced, (ii) which spine axiom or definition of Section 2 it instantiates or discharges, and (iii) the single *emergent representational property* that appears only in the composite. The emergent properties are collected in Table 1.

3.1 Rung i: the possibility of disciplined translation

Part I introduces the primitives themselves: $\mathsf{Dom}, \mathsf{Math}_d, \mathsf{Phys}$, the representation entry (M, P, τ, σ), the prestack/stack, the pipeline (2), and the status system. There is as yet nothing to compose; the emergent contribution of the base rung is *the very possibility of a disciplined mathematics↔physics translation*: syntax (the mathematics), semantics (the physical role), and epistemic warrant (the status) are, for the first time, tracked as *data* rather than prose. This is what makes the later emergences *sayable*.

3.2 Rung i → ii: coalgebraic anatomy

Part II makes Real_α concrete. The abstract-information step becomes $\Phi(X) = \mathrm{Mot}(X)$; the realization channels become the honest functors $\mathrm{Real}_\alpha \in \{\text{Betti}, \text{de Rham}, \text{Hodge}, \text{étale}\}$ with $H_\alpha(X) \simeq \mathrm{Real}_\alpha(\mathrm{Mot}(X))$; and Obs becomes the period $\mathrm{per}(\Pi) = \int_\Gamma \omega$ of a period datum $\Pi = (X, D, \omega, \Gamma)$. The master refinement is

$$\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}}), \qquad \Delta(\mathcal{A}^{\mathfrak{m}}) = \textstyle\sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}, \qquad \delta : \mathcal{L} \to \mathcal{L} \wedge \mathcal{L}, \tag{3}$$

with $\mathcal{A}^{\mathfrak{m}}$ a motivic amplitude object in a comodule over a Hopf algebra of motivic periods. Part II's central proposition (II/T1) is that a monoidal realization functor transports the coaction: $\mathrm{Real}_\alpha(\Delta(a)) = (\mathrm{Real}_\alpha \otimes \mathrm{Real}_\alpha)(\Delta(a))$, so the decomposition of $\mathcal{A}^{\mathfrak{m}}$ descends to a decomposition of the analytic/numerical amplitude. This *instantiates* Axiom 4 (Decomposition) exactly.

Emergent property. Composing I with II endows observables with an internal *coalgebraic anatomy*: an amplitude is no longer an opaque number but a structured object that *decomposes* into primitive informational pieces (weight-graded polylogarithmic generators, II/T2), and this anatomy is stable under any compatible realization channel. Numerical measurement and structural decomposition become two views of one object. Status: the transport is S/H (formal for comodules; physical only when the amplitude admits a motivic lift), with the elliptic/non-Tate obstruction (II/T4) a genuine S limitation theorem and the cosmic Galois functoriality (II/T5) explicitly H/P.

3.3 Rung ii $\to$ iii: geometric variation

A period datum requires X to be an *actual* algebraic/analytic space, and a family of amplitudes over kinematic space is a *variation* of that data. Part III supplies the stage: schemes and $\operatorname{Spec} R$, the on-shell observable algebra R/I, moduli spaces and stacks, and, crucially, variations of Hodge structure with the Gauss–Manin connection. Part III/T1 upgrades II's per-object pipeline to a *family* pipeline: the assignment $s \mapsto \mathrm{Real}_{\mathrm{Hodge}}(\mathrm{Mot}(X_s))$ is a flat section of a variation of Hodge structure over the base S, and the periods satisfy the Picard–Fuchs equation $\nabla\,\Pi(s) = 0$. Positive geometries add a second, geometric decomposition principle: the residue of a canonical form along a boundary is the canonical form of the boundary geometry (III/T3), conjecturally isomorphic to II's coaction tree.

Emergent property. The composite exhibits *geometric variation of physical possibility*: motives and periods are now understood as living over and varying across concrete moduli, so "physical possibility" acquires a geometric object: the moduli space itself, its singularities (thresholds), and its compactification (asymptotic states). Amplitudes become flat sections; analytic continuation becomes monodromy; a differential equation (Picard–Fuchs) governs the whole family. The decomposition axiom now has *two* independent geometric realizations (motivic coaction and positive-geometry residue), whose conjectured coincidence (III/T3, status H) is the programme's most interesting open problem (Section 10).

3.4 Rung iii $\to$ iv: functorial conservation

Geometric spaces carry topological invariants stable under continuous deformation. Part IV supplies (co)homology, characteristic classes, bordism, and, in the decisive step, TQFT as a symmetric monoidal functor $Z : \mathsf{Bord}_n \to \mathsf{Vect}$. Part IV/T1 identifies Z as a bona fide representation entry $(\mathsf{Bord}_n, \mathsf{Vect}, Z, \mathsf{S})$ in the sense of Theorem 2.1, with Φ sending a physical scenario to the underlying closed $(n-1)$-manifold (its boundary object) on which the theory acts, $\mathrm{Real}_\alpha = Z$, and Obs the dimension/trace; the symmetric monoidal (covariant) functoriality $Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2)$, $Z(M_1 \circ M_2) = Z(M_1) \circ Z(M_2)$ is precisely Axiom 4 for disjoint unions and gluing. The Atiyah–Singer index theorem $\mathrm{ind}(D) = \int_X \hat{A}(X) \wedge \mathrm{ch}(E)$ (IV/T3) is an unimpeachably S instance of the master formula observable = Real(abstract structure).

Emergent property. The composite yields *functorial conservation*: realization becomes a functor over cobordisms, so "the same physics" is now transported *coherently* along histories, and topological invariants (charges, anomalies, phases) emerge as data that *no local deformation can change*. This is also the first appearance of category theory as the ambient grammar: TQFT is the historically first fully rigorous, status-S instance of the pipeline as a functor, and the cobordism hypothesis (IV/T2), which classifies extended TQFTs by their value on a point, announces the extreme compositional rigidity that Part V will abstract.

3.5 Rung iv → v: universal compositional grammar

Part IV *exhibited* one rigorous "physics as a functor." Part V supplies the *general theory* in which that instance and every other realization functor lives: categories, functors, natural transformations, monoidal and dagger-compact structure, and the homotopy-type-theoretic layer of identity types and univalence. Two discharges occur here. First, V/T2 (Curry–Howard–Lambek) gives a categorical semantics for the whole pipeline as a single composite functor $\mathsf{TypeTheory} \to \mathsf{Phys}$, simultaneously interpreting logic, code, and physical representation. Second, and centrally, V/T1 turns Axiom 2 from a *postulate* into a *theorem*: univalence $(A = B) \simeq (A \simeq B)$ forces any internally definable realization to identify equivalent structures. Dagger-compact categories additionally give a self-contained categorical no-cloning theorem (V/T3), which will *force* Part VI's encoding-based definition of high availability.

Emergent property. The composite makes the implicit functoriality of IV into an *explicit universal grammar of composition and identity.* Equivalence of representations is now *provably* respected (univalence), not merely stipulated; the gap between gauge-dependent and gauge-invariant quantities becomes the precise gap between externally and internally definable realizations. This is the rung at which the library becomes self-describing: its own translations are objects of the theory it studies.

3.6 Rung v → vi: gauge / fault-tolerant high availability

Part VI is the capstone. Using V's fibered categories and groupoid quotients it *defines rigorously* what Part I merely posited: a stack is a fibered category satisfying descent for a Grothendieck topology, and the quotient stack $[X/G]$ retains isotropy, $\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x)$. It then unifies two families of "many descriptions, one content" phenomena: gauge redundancy $[X/G]$ (the coarse quotient X/G forgets the stabilizers that carry anomaly and path-integral weight), and quantum high-availability encodings $\iota : \mathcal{H}_{\log} \hookrightarrow \mathcal{H}_{\mathrm{phys}}$ with the subsystem decomposition

$$\mathcal{H}_{\mathrm{phys}} \;\simeq\; (\mathcal{H}_{\log} \otimes \mathcal{H}_{\mathrm{gauge}}) \;\oplus\; \mathcal{H}_{\mathrm{err}}. \tag{4}$$

For surface codes (VI/T3), logical operators are the nontrivial homology classes $H_1(\Sigma; \mathbb{Z}_2)$ of the code surface, a *literal*, status-S reuse of Part IV's homology dictionary entry. Renormalization reappears as the antipode of the Connes–Kreimer Hopf algebra (VI/T4), reusing II's coalgebra on a different combinatorial substrate.

Emergent property. The composite delivers *high availability / fault tolerance*: gauge/quotient constructions and quantum error-correcting codes are one representational phenomenon, in which many physical representatives realize one logical/observable content, robustly against a controlled class of variations. This is the property the entire ladder was climbing toward, and its two faces (gauge redundancy, quantum encoding) are identified by the isotropy correspondence at the heart of the Closure Theorem (Section 6).

Rung	New structure	Spine axiom touched	Emergent property (composition adds)
I	Entries (M, P, τ, σ); prestack/stack; pipeline; $\mathsf{S/H/P}$	all four posited	Disciplined translation: syntax vs. semantics vs. status, tracked as data
II	Motives; period data; $\mathcal{A}^{\mathfrak{m}}$; coaction Δ, cobracket δ	Axiom 4 instantiated	*Coalgebraic anatomy*: observables decompose into primitive pieces
III	Schemes; moduli stacks; VHS; Gauss–Manin; positive geometries	Axioms 3 and 4 refined	*Geometric variation*: possibilities live over and vary across moduli
IV	(Co)homology; characteristic classes; bordism; TQFT $Z : \mathsf{Bord}_n \to \mathsf{Vect}$	Axioms 1 and 4 realized	*Functorial conservation*: invariants stable under deformation; realization is a functor
V	Functors; naturality; monoidal/dagger; identity types; univalence	Axiom 2 *proved*	*Universal compositional grammar*: equivalence provably respected
VI	Sheaves; stacks; $[X/G]$; QEC codes; Hopf renormalization	Axioms 2 and 3 *discharged*	*High availability / fault tolerance*: many representatives, one content

Table 1: The modular composition ladder. Each rung introduces one new structure, touches one or more spine axioms of Section 2, and contributes one emergent representational property that exists only in the composite.

4 A unified mathematical framework

Having traversed the ladder informally, we now give the single mathematical object in which all six faculties sit, and the master commuting diagram that ties them together. We stress once more (Section 1) that "unified" here means *assembled from modular parts by a canonical construction*, not *fused into one primitive.*

4.1 The total representation category

Definition 4.1 (Total representation category)**.** Let $\mathsf{Rep}_{\text{phys}} : \mathsf{Dom}^{\text{op}} \to \mathsf{Gpd}$ be the representation prestack (Theorem 2.3). Its *Grothendieck construction* $\int_{\mathsf{Dom}} \mathsf{Rep}_{\text{phys}}$ is the category whose objects are pairs (d, E) with $d \in \mathsf{Dom}$ and $E \in \mathsf{Rep}_{\text{phys}}(d)$ a representation entry over d, and whose morphisms $(d, E) \to (d', E')$ are pairs (u, ϕ) with $u : d \to d'$ in Dom and $\phi : E \to u^*E'$ an isomorphism in $\mathsf{Rep}_{\text{phys}}(d)$. The projection $\pi : \int_{\mathsf{Dom}} \mathsf{Rep}_{\text{phys}} \to \mathsf{Dom}$, $(d, E) \mapsto d$, is a fibered category (a Grothendieck fibration in groupoids), and $\mathsf{Rep}_{\text{phys}}$ is recovered as its groupoid of sections.

The Grothendieck construction is the categorical device that turns the domain-indexed, *local* data of the library into a single *global* category without collapsing the domain index, which is exactly the modular discipline of Section 1 made precise. The six modules populate the fibers: $\mathsf{Rep}_{\mathrm{phys}}$(motives), $\mathsf{Rep}_{\mathrm{phys}}$(alg. geom.), $\mathsf{Rep}_{\mathrm{phys}}$(alg. top.), $\mathsf{Rep}_{\mathrm{phys}}$(cat/HoTT), $\mathsf{Rep}_{\mathrm{phys}}$(sheaf/stack) are the fibers over the corresponding domains.

4.2 The global realization functor

Each module supplies, over its fiber, a factorization of the pipeline (2) through named categories. Writing $\mathsf{Info}_d, \mathsf{Rep}_d, \mathsf{Meas}_d$ for the abstract-information, physical-representation and measurement categories of domain d, the module's content is a triple of functors

$$\mathsf{Math}_d \xrightarrow{\Phi_d} \mathsf{Info}_d \xrightarrow{\mathrm{Real}_\alpha^{(d)}} \mathsf{Rep}_d \xrightarrow{\mathrm{Obs}_d} \mathsf{Meas}_d,$$

whose composite $\mathrm{Obs}_d \circ \mathrm{Real}_\alpha^{(d)} \circ \Phi_d$ is a functor $\mathsf{Math}_d \to \mathsf{Meas}_d \subseteq \mathsf{Phys}$ whenever each factor is functorial (Part I/T1). The following are the six instances, each pinned to its status:

Module	Φ_d	$\mathrm{Real}_\alpha^{(d)}$	Obs_d
II	$X \mapsto \mathrm{Mot}(X)$	Betti/de Rham/Hodge/étale	period $\mathrm{per}(\Pi) = \int_\Gamma \omega$
III	family $X_s \mapsto$ VHS	Gauss–Manin flat transport	flat section / monodromy
IV	closed $(n-1)$-manifold	TQFT functor $Z : \mathsf{Bord}_n \to \mathsf{Vect}$	dimension / trace / index
V	type $A : \mathcal{U}$	internal construction	term extraction / global sections
VI	logical data $\mathcal{H}_{\log}$	encoding $\iota : \mathcal{H}_{\log} \hookrightarrow \mathcal{H}_{\mathrm{phys}}$	syndrome / logical readout

By the status calculus (Section 2.4), the status of a *composite* realization is the max over its three factors: a module's realization is status-S only when all three functors are honestly constructed (as for TQFT), and degrades to H or P as soon as any factor is merely interpretive or speculative.

4.3 The master commuting diagram

Figure 2 assembles the six faculties. The left column is the ladder: the builds-on functors $J_{d,d+1} : \mathsf{Math}_d \to \mathsf{Math}_{d+1}$ realize each module as living inside the next (motivic amplitude objects are objects of a derived category of motives of a variety of III; the moduli space of III has the (co)homology of IV; TQFT is an instance of V's functor grammar; V's fibered categories are the raw material of VI's stacks). Every module realizes into the single target Phys through its global realization functor Real_d, and the triangles commute because realization is compatible with the builds-on functors: realizing at a higher level and then restricting agrees with realizing at the lower level. The dashed arrow closes the ladder: Part VI's stackification proves that the top of the tower, Part I's representation stack, exists (Section 6).

$$
\begin{tikzcd}
\mathsf{Math}_{\mathrm{I}} \arrow[r, "\mathrm{Real}_{\mathrm{I}}"] \arrow[d, "J_{12}"'] & \mathsf{Phys} \\
\mathsf{Math}_{\mathrm{II}} \arrow[ur, "\mathrm{Real}_{\mathrm{II}}"] \arrow[d, "J_{23}"'] & \\
\mathsf{Math}_{\mathrm{III}} \arrow[uur, "\mathrm{Real}_{\mathrm{III}}"] \arrow[d, "J_{34}"'] & \\
\mathsf{Math}_{\mathrm{IV}} \arrow[uuur, "\mathrm{Real}_{\mathrm{IV}}"] \arrow[d, "J_{45}"'] & \\
\mathsf{Math}_{\mathrm{V}} \arrow[uuuur, "\mathrm{Real}_{\mathrm{V}}"] \arrow[d, "J_{56}"'] & \\
\mathsf{Math}_{\mathrm{VI}} \arrow[uuuuur, "\mathrm{Real}_{\mathrm{VI}}"'] \arrow[uuuuu, bend right=60, dashed, "\substack{\text{stackify}\\ \text{(closure, T2)}}"']
\end{tikzcd}
$$

Figure 2: The master commuting diagram of the library. Vertical: the builds-on functors $J_{d,d+1}$ (the ladder). Diagonal: the global realization functors $\mathrm{Real}_d : \mathsf{Math}_d \to \mathsf{Phys}$, forming a cocone over the single physical target Phys; each triangle $\mathrm{Real}_{d+1} \circ J_{d,d+1} \Rightarrow \mathrm{Real}_d$ commutes up to coherent isomorphism (realization is compatible with refinement). Dashed: Part VI's stackification carries the final rung ($\mathsf{Math}_{\mathrm{VI}}$) back to Part I, closing the ladder (Theorem 6.1).

4.4 The master formulas as one system

The programme has a small set of recurring formulas. In the unified framework they are one system, read off the total category and its fibers:

$$\llbracket M \rrbracket_{\mathrm{phys}} := \text{physical representation of } M \qquad \text{(bracket; Theorem 2.1)} \tag{5}$$

$$\mathrm{Obs}_\alpha(M) = \mathrm{Obs}(\mathrm{Real}_\alpha(\Phi(M))) \qquad \text{(pipeline; Axiom 1)} \tag{6}$$

$$\text{observable} = \mathrm{Real}(\text{abstract structure}) \qquad \text{(master slogan)} \tag{7}$$

$$\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}}) \qquad \text{(motivic pre-numericality; II)} \tag{8}$$

$$\Delta(\mathcal{A}^{\mathfrak{m}}) = \textstyle\sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2} \qquad \text{(coaction / full anatomy; II,VI)} \tag{9}$$

$$\delta : \mathcal{L} \to \mathcal{L} \wedge \mathcal{L} \qquad \text{(cobracket / primitive split; II)} \tag{10}$$

$$\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x) \qquad \text{(isotropy retention; I,III,V,VI)} \tag{11}$$

$$\mathcal{H}_{\mathrm{phys}} \simeq (\mathcal{H}_{\mathrm{log}} \otimes \mathcal{H}_{\mathrm{gauge}}) \;\oplus\; \mathcal{H}_{\mathrm{err}} \qquad \text{(high-availability decomposition; VI)} \tag{12}$$

Formulas (5)–(7) are the spine; (8)–(10) are its decomposition refinement; and (11)–(12) are the isotropy and high-availability pair that the Closure Theorem identifies. Table 2 records how each module's central theorem is a special case of one of these formulas.

Module theorem	Master formula	Sense of specialization
I/T1 (pipeline is a functor)	(6)	functoriality of $\mathrm{Obs} \circ \mathrm{Real}_\alpha \circ \Phi$
II/T1 (coaction transport)	(9)	Real_α monoidal $\Rightarrow$ carries Δ
II/T2 (weight primitives)	(10)	δ generates $\mathcal{L}$ from weight-1
III/T1 (Gauss–Manin)	(6)	pipeline fibered over moduli, flat section
III/T3 (positive-geometry residue)	(9)	residue tree $\cong$ coaction tree (conj., H)
IV/T1 (TQFT is an entry)	(6),(7)	$Z = \mathrm{Real}_\alpha$, Obs = trace
IV/T3 (index theorem)	(7)	$\mathrm{ind}(D) = \mathrm{Real}_{\mathrm{top}}(\text{char. classes})$
V/T1 (univalence)	(11)	equivalence = identity for internal Real
VI/T1 (stacks retain isotropy)	(11)	$\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x)$
VI/T3 (surface code)	(12)	$\mathcal{H}_{\log} \cong H_1(\Sigma; \mathbb{Z}_2)$
VI/T4 (Connes–Kreimer antipode)	(9)	Δ on graphs, antipode = counterterm

Table 2: Each module's central theorem as a special case of a master formula (5)–(12). This is the quantitative content of "the whole machinery is recovered module by module."

5 Cross-cutting themes

Certain items recur across two or more modules; these are the load-bearing joints of the synthesis. We treat five.

5.1 One realization formula, six instances

The pipeline (6) is the single backbone formula, instantiated concretely by every module (the $\Phi_d, \mathrm{Real}_\alpha^{(d)}, \mathrm{Obs}_d$ table of Section 4). The instances are not analogies: in IV the realization functor is literally a TQFT; in II it is literally a period; in VI it is literally a quantum encoding. What varies is only the epistemic status of the composite, governed by the status calculus. This is why the library can be simultaneously ambitious (one formula for all of physical representation) and honest (each instance is graded).

5.2 One Hopf/coalgebra of decomposition

Axiom 4 (Decomposition) has a strikingly uniform incarnation. Every module supplies a decomposition operation, and all are instances of a single algebraic pattern: a *graded connected Hopf/coalgebra structure with a primitive-antisymmetric part.*

Principle 5.1 (Universal decomposition). The decomposition operations of the library — the motivic coaction Δ and Goncharov cobracket δ (II); the positive-geometry residue (III); the boundary operator ∂ and the long exact/spectral sequences (IV); and the Connes–Kreimer coproduct with antipode S (VI) — are all realizations of one graded, connected coalgebra-with-primitives, differing only in the combinatorial data they act on (polylogarithm symbols, boundary strata, chains, rooted trees).

Two facts make Theorem 5.1 more than a slogan. First, II and VI literally share references: Connes–Kreimer [22] and Goncharov [16] appear in both reference pools, and both are graded connected Hopf algebras with antipode: one on rooted trees (subdivergences), one on polylogarithm symbols (transcendentality). Second, the antipode formula of VI/T4, $S(\Gamma) = -\Gamma - \sum S(\Gamma')\,(\Gamma/\Gamma')$, is the exact algebraic dual of the recursive residue of III/T3 and of the iterated coaction of II: subtraction of subdivergences, restriction to boundary strata, and extraction of primitive pieces are one operation appearing in three forms. The conjectural equality of the motivic coaction tree and the positive-geometry residue tree (III/T3) is the sharpest open instance of this principle (Section 10), and we flag it status H.

5.3 A functoriality gradient of increasing concreteness

Functoriality of realization threads the ladder at three increasing levels of concreteness: Part I *posits* it (Axiom 1, Real_α implicitly functorial); Part IV *exhibits* the first fully rigorous instance (TQFT $Z : \mathsf{Bord}_n \to \mathsf{Vect}$, IV/T1, status S); Part V supplies the *general theory* (functors, natural transformations, the whole grammar, V/T2). The cobordism hypothesis (IV/T2) is the historical hinge: it is a topological, status-S theorem whose proof *requires* the higher-categorical duality data that Part V formalizes, so it simultaneously closes Part IV and motivates Part V.

5.4 The isotropy theorem, four proofs

The single most recurrent theorem of the library is the retention of automorphism (stabilizer) data by the stack quotient, (11). It appears *four* times, at strictly increasing rigor:

1. **Part i (informal).** The representation stack (Theorem 2.4) glues entries "not by equality but by equivalence," with automorphisms modeling gauge and duality, a posited property.

2. **Part iii (algebro-geometric, iii/T2).** If x has stabilizer $\mathrm{Stab}_G(x) \neq 1$, the coarse map $[X/G] \to X/G$ is not an isomorphism near x; the local structure of $[X/G]$ at x is $[\operatorname{Spec} \mathcal{O}_{X,x}/\operatorname{Stab}_G(x)]$, recording exactly the data lost by X/G.

3. **Part v (type-theoretic, v/T4).** The set quotient X/G (a 0-type) forgets the path (automorphism) data that the groupoid quotient $X/\!\!/G$ (a 1-type when stabilizers are discrete) retains as nontrivial loops; passing $X \rightsquigarrow [X/G]$ is passing from a 0-type to a 1-type in the truncation hierarchy.

4. **Part vi (descent-theoretic, vi/T1).** The local structure of $[X/G]$ at x is the classifying stack $[*/\operatorname{Stab}_G(x)]$, whose every point has automorphism group $\mathrm{Stab}_G(x)$; the coarse-space functor collapses Aut to the trivial group, so any observable counting automorphisms (orbifold Euler characteristic, $1/|\operatorname{Stab}|$ path-integral weight) is computable from $[X/G]$ but not from X/G.

These are not four theorems but one theorem with four proofs, and the synthesis presents them as such: this is the axis of the Closure Theorem (Section 6).

5.5 Homology reused literally

Part IV's abstract dictionary entry "homology $H_n(X) \leftrightarrow$ conserved extended structures" becomes, in Part VI, a concrete quantum-information mechanism: for a surface code on a genus-g surface Σ, the logical operators are the generators of $H_1(\Sigma; \mathbb{Z}_2)$ and the code space is $(\mathbb{C}^2)^{\otimes 2g}$ (VI/T3). Contractible error strings are boundaries and act trivially, which is exactly the Part IV entry "boundary $c = \partial b \leftrightarrow$ pure gauge / physically null sector." This is direct literal reuse, not analogy, and it is the strongest (status-S) bridge in the entire library between algebraic topology, gauge/stacks, and quantum information.

6 The Closure Theorem

We now state and assemble the main result of the synthesis. It has three faces: a descent-theoretic statement that the representation stack *exists*, an isotropy statement that unifies its four proofs, and a physical statement that identifies gauge redundancy with quantum high availability.

6.1 Statement

Theorem 6.1 (Closure Theorem)**.** *Let* (Dom, J) *be the site of mathematical domains and* $\mathsf{Rep}_{\mathrm{phys}} : \mathsf{Dom}^{\mathrm{op}} \to \mathsf{Gpd}$ *the representation prestack of Part* I *(Theorem 2.3), whose local data over the domains* $\{motives, alg.\ geom., alg.\ top., cat/HoTT, sheaf/stack\}$ *are supplied concretely by Parts* II–VI *(periods and coactions; VHS and Gauss–Manin; (co)homology and TQFT functors; identity data and univalence; encodings and stabilizers). Then:*

1. (Existence, status S/H.) *There is a stack* $\mathsf{Rep}^J_{\mathrm{phys}}$ *on* (Dom, J) *and a universal morphism* $\eta : \mathsf{Rep}_{\mathrm{phys}} \to \mathsf{Rep}^J_{\mathrm{phys}}$ *such that for every stack* $\mathcal{G}$*, precomposition*

$$\eta^* : \ \underline{\mathrm{Hom}}_{\mathrm{Stack}}(\mathsf{Rep}^J_{\mathrm{phys}}, \mathcal{G}) \ \xrightarrow{\ \simeq\ } \ \underline{\mathrm{Hom}}_{\mathrm{CFG}}(\mathsf{Rep}_{\mathrm{phys}}, \mathcal{G})$$

   is an equivalence of groupoids (Part VI/T2)*. Consequently Part* I*'s "representation stack"* (Theorem 2.4) *exists and equals* $\mathsf{Rep}^J_{\mathrm{phys}}$*: when the local realization data of Parts* II–V *agree on overlaps they assemble into a global object of* $\mathsf{Rep}^J_{\mathrm{phys}}$*, exactly satisfying the informal gluing of Theorem 2.4 and Axiom 3.*

2. (Closure under the six operations, status S.) $\mathsf{Rep}^J_{\mathrm{phys}}$ *is closed under the five library operations of Theorem 2.2 — restriction, transport, composition, decomposition, realization — interpreted through the six faculties: restriction is* J*-refinement; transport is univalence-respecting equivalence* (V/T1)*; composition is functor composition with status* max*-propagation* (I/T3)*; decomposition is the universal coalgebra/coaction of Theorem 5.1* (II,III,IV,VI)*; and realization is the global functor of Section 4* (IV)*.*

3. (Isotropy keystone, status S.) *For every gauge datum* (X, G) *occurring in a fiber and every* $x \in X$*,*

$$\mathrm{Aut}_{\mathsf{Rep}^J_{\mathrm{phys}}}(x) \ \simeq \ \mathrm{Aut}_{[X/G]}(x) \ \simeq \ \mathrm{Stab}_G(x),$$

a single isomorphism established four independent ways (I,III/T2, V/T4,VI/T1; Section 5.4)*. Its physical face is the equivalence of* gauge redundancy *and* quantum high availability*: the encoding $\iota : \mathcal{H}_{\log} \hookrightarrow \mathcal{H}_{\text{phys}}$ with $\mathcal{H}_{\text{phys}} \simeq (\mathcal{H}_{\log} \otimes \mathcal{H}_{\text{gauge}}) \oplus \mathcal{H}_{\text{err}}$* (12) *presents the same "many representatives, one logical content" structure as the quotient stack, with $\mathcal{H}_{\text{gauge}}$ the isotropy/gauge directions and $\mathcal{H}_{\text{err}}$ the syndrome sectors* (VI/T3)*.*

Assembly. Part (1) is Part VI/T2 verbatim: one adapts the plus construction from set-valued to groupoid-valued coefficients, setting $\mathsf{Rep}^{+}_{\text{phys}}(U) = \varinjlim_{\{U_i \to U\} \in J(U)} \operatorname{Desc}(\{U_i \to U\}, \mathsf{Rep}_{\text{phys}})$ over the filtered category of covers of U (any two covers admit a common refinement by pairwise pullbacks), and $\mathsf{Rep}^{J}_{\text{phys}} := \mathsf{Rep}^{++}_{\text{phys}}$; then $\mathsf{Rep}^{+}_{\text{phys}}$ is a prestack for any fibered category in groupoids and a stack when $\mathsf{Rep}_{\text{phys}}$ is already a prestack, and the universal property follows because descent in the target stack $\mathcal{G}$ makes every descent datum effective, yielding the essentially unique factorization through η. This is the general stackification theorem for fibered categories over a site [13, 12], specialized to $\mathsf{Rep}_{\text{phys}}$; the mathematics is status-S, and only the modeling claim that the Parts II–V realization data *is* a J-descent datum is status-H (Section 9).

Part (2) reads off the five closure operations from the six modules. Closure under restriction and realization is definitional for a stack. Closure under transport is Part V/T1: an internally definable realization cannot distinguish equivalent $A \simeq B$, since univalence gives a path $A = B$ and path induction (the J-rule) transports every construction; hence equivalence-related entries have equal image. Closure under composition, with status tracked, is Part I/T3: the composite of two entries has status $\max(\sigma_1, \sigma_2)$ under $\mathsf{S} \preceq \mathsf{H} \preceq \mathsf{P}$, so $\mathsf{Rep}^{J}_{\text{phys}}$ is closed as a status-graded category (a lax monoidal functor to $(\{\mathsf{S}, \mathsf{H}, \mathsf{P}\}, \max)$). Closure under decomposition is Theorem 5.1: each fiber carries its coalgebra of decomposition (motivic coaction, residue, boundary, antipode), and these are compatible with realization by the monoidality of II/T1 and the symmetric-monoidal gluing of the TQFT of IV/T1.

Part (3) is the isotropy correspondence of Section 5.4, whose four proofs agree, together with its physical face. The stack $\mathsf{Rep}^{J}_{\text{phys}}$ retains automorphisms by construction (it is groupoid-valued), and for a gauge datum (X, G) the fiber automorphism group at x is $\operatorname{Stab}_G(x)$ by Part VI/T1. The identification with quantum high availability is Part VI's Remark on the sheaf/stack/code analogy made literal by Part VI/T3: a stabilizer code stores logical data across physical degrees of freedom together with transformations (stabilizers, gauge automorphisms) that do not change the logical content, so the code space is the "logical fiber" of an encoding stack, with $\operatorname{Stab}_G(x) \leftrightarrow$ stabilizer group $\leftrightarrow \mathcal{H}_{\text{gauge}}$ and error sectors $\leftrightarrow \mathcal{H}_{\text{err}}$. For surface codes this identification is exact and status-S (logical operators $= H_1(\Sigma; \mathbb{Z}_2)$); the general slogan "gauge redundancy is to physics what fault-tolerant encoding is to quantum information" is status-H (Section 9, Section 10). □

Corollary 6.2 (The ladder closes)**.** *The six modules are not merely an expository decomposition. Theorem 6.1(1) shows that Part* I*'s* opening *definition (the representation stack) is a* theorem *provable from Part* VI*'s descent machinery applied to the concrete data of Parts* II–V*. The ladder* I $\to \cdots \to$ VI *therefore closes back onto Part* I*: what was posited at the base is discharged at the capstone.*

$$\begin{tikzcd}
& \boxed{\text{Part I: } \mathsf{Rep}_{\mathrm{phys}} \text{ posited stack-like}} \arrow[d, "\text{deferred}"] & \\
& \boxed{\text{Parts II–V: local realization data}} \arrow[d, "\text{descent datum}"] & \\
\boxed{\text{gauge redundancy } [X/G]} \arrow[r, "\mathrm{Stab}_G(x)", "\simeq"'] & \boxed{\text{Part VI: stackify } \mathsf{Rep}^J_{\mathrm{phys}}} \arrow[uu, dashed, bend left, "\text{closes}"] \arrow[r, "\mathcal{H}_{\mathrm{gauge}}", "\simeq"'] & \boxed{\text{quantum HA } \iota : \mathcal{H}_{\log} \hookrightarrow \mathcal{H}_{\mathrm{phys}}}
\end{tikzcd}$$

Figure 3: The closure loop and its keystone. Part I posits and defers; Parts II–V supply the local data as a descent datum; Part VI stackifies to produce $\mathsf{Rep}^J_{\mathrm{phys}}$, which *is* Part I's stack (dashed "closes" arrow). The isotropy group $\mathrm{Stab}_G(x)$ identifies gauge redundancy (left) with quantum high availability (right): $\mathrm{Stab}_G(x) \simeq \mathcal{H}_{\mathrm{gauge}}$.

6.2 Why the keystone is forced, not chosen

Part V/T3 (categorical no-cloning) explains why the quantum face of the keystone must be an *encoding*, not a *copy*. In a dagger-compact category with biproducts there is no natural diagonal $\Delta : A \to A \otimes A$ compatible with the dagger-compact structure unless A is classical (a special commutative dagger-Frobenius algebra). Hence classical high availability by *replication* has no quantum analogue; the only route to "many physical representatives, one logical content" in the quantum setting is a genuine embedding $\iota : \mathcal{H}_{\log} \hookrightarrow \mathcal{H}_{\mathrm{phys}}$ into a larger, redundant system, which is exactly the gauge/stack structure of (11). Part V thus supplies the categorical *reason* that Part VI's encoding-based definition is the correct one, and the isotropy keystone is forced.

7 Emergence, recomposed

We can now state precisely what "emergence from composition" means in this programme, and check that the whole spine is recovered.

Principle 7.1 (Emergence is compositional, not fundamental). Each emergent property of Table 1 — coalgebraic anatomy, geometric variation, functorial conservation, universal grammar, high availability — is a property of a *composite* of modules that is possessed by *no single module* in isolation. Coalgebraic anatomy needs both the pipeline (I) and the coaction (II); geometric variation needs both the coaction (II) and the moduli (III); functorial conservation needs both the moduli (III) and the topological invariants (IV); and so on. The properties are therefore genuinely emergent in the technical sense of arising at, and only at, a level of the composition hierarchy.

The recomposition is complete in the following sense. Table 2 exhibits every module's central theorem as a special case of a master formula; Theorem 6.1 shows the whole prestack of Part I is recovered as the stackification of the concrete data of Parts II–VI; and Theorem 6.2 shows the recovery is *exact*, in that Part I's own definition is satisfied. Thus the informally posited Level-I structure is not merely *illustrated* by the six modules; it is *constructed* by

Module (principal dictionary)	S	S/H	H	H/P	Total
I (core translation)	9	5	2	0	16
II (motivic / amplitude)	20	9	2	1	32
III (algebraic geometry)	35	13	4	0	52
IV (algebraic topology)	46	4	3	0	53
V (category theory / HoTT)	26	31	18	0	75
VI (Lie/Hopf/sheaf/stack)	26	12	2	0	40
Total	**162**	**74**	**31**	**1**	**268**

Table 3: Status-label census of the six principal dictionaries. Of 268 entries, 162 (60.4%) are standard S, 74 (27.6%) are S/H (standard as mathematics, heuristic in physical reading), 31 (11.6%) are heuristic H, and a single entry (II's cosmic Galois group) is H/P. No entry is purely P: the library's speculative content lives in its *theorems and conjecture*, not its dictionary.

them. This is the precise content of the claim that the library is a genuine hierarchical composition rather than a taxonomy.

8 A quantitative status-label census

The status system records how much epistemic warrant each entry carries; here we make it quantitative. Table 3 tabulates the status distribution of the principal math→physics dictionary table of each module (the shared cross-cutting operations and domains-as-faculties tables of Part I are excluded to avoid double counting; the Part VI quantum-information cross-dictionary is excluded because its entries carry interpretive rather than S/H/P labels). Composite labels are counted in their own columns.

Table 3 brings out two contrasts. Part IV (algebraic topology) is the most *standard* module (87% S): TQFT, index theory, and characteristic classes are established mathematical physics. Part V (category theory / HoTT) is the most *interpretive* (65% S/H or H): its mathematics is standard but its physical readings (a type as a space of states, a functor as a physical theory) are heuristic by construction. That fits the module's character: the abstract compositional grammar is where physical interpretation is most optional.

Status of the candidate theorems. The library's roughly thirty candidate theorems partition by status as follows. *Standard* (S): I/T1–T2, II/T1–T3, III/T1–T2,T4, all of IV/T1–T4, all of V/T1–T4, VI/T1–T4. These are either established results reframed (Atiyah–Singer, Lurie's cobordism hypothesis, Giraud/Vistoli stackification, Connes–Kreimer, univalence) or immediate consequences. *Original heuristic* (H): I/T3 (the status calculus, original to the programme), III/T3 (the positive-geometry/coaction isomorphism, conjectural), and the general slogan "gauge redundancy ≅ fault-tolerant encoding" beyond surface codes. *Speculative* (P or H/P): II/T5 (functoriality of the cosmic Galois group action) and the overarching

Conjecture of Section 10. The Closure Theorem (Theorem 6.1) is S in its mathematics and H in the modeling claim that identifies the physical realization data with a descent datum.

9 Limitations

The programme states five limitations, which we reproduce verbatim and then discuss module by module.

L1. Not every mathematical analogy is a physical theory — status labels are part of the formalism, not decoration.

L2. Motives are NOT literally "the functions of" an object in standard mathematics; that reading is this project's proposed semantic layer (status H), to be clearly flagged in every module that uses it.

L3. A single universal functor from all of mathematics to all of physics is unlikely to be well-defined — the representation-stack approach is deliberately local/domain-indexed. Modules should not overclaim a single functor $\mathsf{Math} \to \mathsf{Phys}$.

L4. Not every physical observable is known to be a period; some amplitudes require elliptic, modular, non-Tate, or more general analytic structures.

L5. Gauge redundancy $\neq$ physical symmetry: gauge symmetry identifies different descriptions of the same physical content; physical symmetry maps one physical state to a genuinely different one.

Discussion. **L1** is honored globally by the status census (Section 8): the labels are carried into every table and into the status calculus (Section 2.4), which propagates them through composition. **L2** constrains Part II: the functional-essence reading $\mathrm{FE}(X) = \mathrm{Mot}(X)$ is flagged H wherever it appears, and the elliptic/non-Tate obstruction (II/T4) is a genuine S limitation theorem, not a caveat. **L3** is the reason the synthesis is *modular*: the unified framework of Section 4 is a Grothendieck construction over a *domain-indexed* prestack, precisely so as *not* to assert a single global functor; the closure of Theorem 6.1 is closure of a *stack*, which lives over the site Dom and never collapses the domain index. **L4** bounds the amplitude claims of Parts II–III: the master formula $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathsf{m}})$ is asserted only for amplitudes admitting motivic lifts, and II/T4 delimits exactly where this fails. **L5** bounds the keystone of Part VI and Theorem 6.1(3): the identification is of gauge *redundancy* (descriptive freedom) with high availability, *not* of physical symmetry with high availability; the isotropy group $\mathrm{Stab}_G(x)$ is a redundancy stabilizer, and physical symmetries, which move between genuinely distinct physical states, are explicitly outside the scope of the keystone.

10 Open problems and the research programme

The synthesis answers the source's own conjecture by listing the testable subprograms it calls for. We state the conjecture and the open problems.

Conjecture 10.1 (Motivic–categorical information ontology, status P)**.** A sufficiently broad class of physical theories admits a representation-stack description in which states, processes, observables, amplitudes, gauge redundancies, and measurement values arise as realizations of structured categorical, homotopical, geometric, and motivic information.

The following are the concrete, citable open problems that would test Theorem 10.1, one per module.

I. A fully rigorous stackification of the *specific* groupoid $\mathsf{Rep}_{\text{phys}}$ (as opposed to the general theorem of VI/T2) is original work; connections to formalizations of "prestacks of physical theories" / "stacks of QFTs" (e.g. factorization homology, or Schreiber's cohesive-topos programme [11]) remain to be made precise.

II. The precise definition of the cosmic Galois group for a general QFT (beyond φ^4) remains conjectural; II/T5 is H/P pending post-2016 results in the Panzer–Schnetz [21] / Brown [20] line.

III. The conjectural isomorphism between positive-geometry residue trees and motivic coaction trees (III/T3) is, to our knowledge, unproven in general; a "Hopf algebra of the amplituhedron" would settle the sharpest instance of Theorem 5.1.

IV. Identifying a *given* SPT phase's bordism invariant (IV/T5) is model-dependent; each lattice-model claim needs its own citation beyond the general Freed–Hopkins classification [34].

V. Concrete cohesive-HoTT formalizations of specific gauge theories (Yang–Mills, Chern–Simons) beyond Schreiber's general framework are largely programmatic.

VI. The general claim "gauge redundancy is to physics what fault-tolerant encoding is to quantum information," beyond surface codes, is H; the holographic quantum-error-correction / entanglement-wedge-reconstruction literature is the natural, more rigorous body of work to strengthen it, and is the single largest opportunity for a genuinely novel contribution.

11 Conclusion

We have recomposed the six modular parts of the library into one coherent account, without collapsing them into a monolith. The representation stack and realization pipeline of Part I are the spine; Parts II–VI instantiate and progressively discharge it; and at each rung of the composition ladder a single emergent property (coalgebraic anatomy, geometric variation, functorial conservation, universal grammar, high availability) arises from the composition and from nowhere else. The Closure Theorem (Theorem 6.1) shows the ladder is genuinely circular: Part I's opening definition is a theorem of Part VI, provable by descent-theoretic stackification of the concrete data of the intervening modules, with the isotropy correspondence $\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x)$ (one theorem, four proofs) as its axis and *gauge redundancy* $\cong$ *quantum high availability* as its keystone.

The programme's four compressed theses survive the synthesis intact, and we adopt them as its closing summary:

> *mathematics is the syntax of physical representation;*
> *motives are functional semantics;*
> *periods are numerical measurements;*
> *coalgebras are decomposition laws.*

To these the ladder adds a fifth, which is the content of the capstone: *stacks are the grammar of redundancy, and redundancy, made robust, is high availability.*

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A Unified operations dictionary

The following cross-cutting table (Part I, Appendix G) records the decomposition and composition operations that recur across all modules, with the physical interpretation each carries. It is the concrete substance of Theorem 5.1.

Operation	Physical interpretation (module)
Boundary ∂	edge / flux / factorization (IV)
Differential d	infinitesimal change (III,IV)
Coboundary d^*	gauge / exact shift (IV,VI)
Cobracket δ	primitive antisymmetric split (II)
Coproduct Δ	full split / coaction (II,VI)
Residue	pole / singular channel (II,III)
Monodromy	analytic continuation (III)
Filtration	scale hierarchy (II,VI)
Grading	charge / degree / loop / weight (II,IV,VI)
Spectral sequence	multi-scale obstruction (IV)
Localization	effective physics at a scale (III)
Completion	perturbation expansion (III)
Blowup	regularize divergence (III)
Quotient	gauge reduction (III,VI)
Pullback	constraint matching (V)
Pushout	couple systems (V)
Trace	partition function / loop amplitude (IV,V)
Dual	measurement / conjugate (V)
Tensor product $\otimes$	parallel composition (IV,V)
Direct sum $\oplus$	superselection (VI)

Operation	Physical interpretation (module)
Hom	space of processes (V)
Ext	bound state / anomaly / deformation (III,IV)
Antipode S	counterterm / renormalization inverse (VI)

B Census methodology

The counts of Table 3 are obtained by tallying the status label of each row of each module's *principal* math→physics dictionary (Part I Appendix A; Part II Appendix B; Part III Appendix C; Part IV Appendix D; Part V Appendix E, category-theory and HoTT tables merged; Part VI Appendix F, Lie/Hopf/sheaf/stack table). Composite labels are recorded in their own columns; a row labeled S/H is counted once in the S/H column, not split. The Part I cross-cutting operations table (Appendix G, reproduced in Section A) and domains-as-faculties table (Appendix H) are excluded to avoid double counting across modules, and the Part VI quantum-information cross-dictionary is excluded because its entries carry interpretive rather than S/H/P labels. These conventions are deterministic and reproducible; the totals are S $= 162$, S/H $= 74$, H $= 31$, H/P $= 1$, summing to 268.

References

[1] M. Long and the YonedaAI Collaboration, *A Math→Physics Representation Library*, Parts I–VI (this series), YonedaAI Research Collective, 2026.

[2] M. F. Atiyah, *Topological quantum field theories*, Publ. Math. IHÉS **68** (1988), 175–186.

[3] J. Lurie, *On the classification of topological field theories*, in *Current Developments in Mathematics 2008*, 129–280; arXiv:0905.0465 (2009).

[4] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies **170**, Princeton University Press, 2009.

[5] J. C. Baez and M. Stay, *Physics, topology, logic and computation: a Rosetta stone*, Lecture Notes in Physics **813** (2010), 95–172; arXiv:0903.0340.

[6] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Graduate Texts in Mathematics **5**, Springer, 1998.

[7] S. Abramsky and B. Coecke, *A categorical semantics of quantum protocols*, Proc. 19th IEEE LiCS (2004); arXiv:quant-ph/0402130.

[8] The Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*, Institute for Advanced Study, 2013.

[9] E. Riehl and D. Verity, *Elements of ∞-Category Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2022.

[10] J. Lambek and P. J. Scott, *Introduction to Higher Order Categorical Logic*, Cambridge Studies in Advanced Mathematics **7**, Cambridge University Press, 1986.

[11] U. Schreiber, *Differential cohomology in a cohesive ∞-topos*, arXiv:1310.7930 (2013).

[12] A. Vistoli, *Notes on Grothendieck topologies, fibered categories and descent theory*, arXiv:math/0412512 (2004).

[13] J. Giraud, *Cohomologie non abélienne*, Grundlehren der mathematischen Wissenschaften **179**, Springer, 1971.

[14] M. Kontsevich and D. Zagier, *Periods*, in *Mathematics Unlimited — 2001 and Beyond*, Springer, 2001, 771–808.

[15] A. B. Goncharov, *Multiple polylogarithms and mixed Tate motives*, arXiv:math/0103059 (2001).

[16] A. B. Goncharov, *Galois symmetries of fundamental groupoids and noncommutative geometry*, Duke Math. J. **128** (2005), 209–284; arXiv:math/0208144.

[17] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, *Classical polylogarithms for amplitudes and Wilson loops*, Phys. Rev. Lett. **105** (2010), 151605; arXiv:1006.5703.

[18] S. Bloch, H. Esnault, and D. Kreimer, *On motives associated to graph polynomials*, Comm. Math. Phys. **267** (2006), 181–225; arXiv:math/0510011.

[19] F. Brown, *Mixed Tate motives over $\mathbb{Z}$*, Annals of Mathematics **175** (2012), 949–976; arXiv:1102.1312.

[20] F. Brown, *Feynman amplitudes, coaction principle, and cosmic Galois group*, arXiv:1512.06409 (2015).

[21] E. Panzer and O. Schnetz, *The Galois coaction on φ^4 periods*, arXiv:1603.04289 (2016).

[22] A. Connes and D. Kreimer, *Renormalization in quantum field theory and the Riemann–Hilbert problem I*, arXiv:hep-th/9912092 (1999).

[23] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer, 1977.

[24] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory*, 3rd ed., Ergebnisse der Mathematik, Springer, 1994.

[25] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHÉS **36** (1969), 75–109.

[26] N. Arkani-Hamed, Y. Bai, and T. Lam, *Positive geometries and canonical forms*, arXiv:1703.04541 (2017).

[27] N. Arkani-Hamed and J. Trnka, *The amplituhedron*, JHEP **10** (2014), 030; arXiv:1312.2007.

[28] K. Costello and O. Gwilliam, *Factorization Algebras in Quantum Field Theory*, Vol. 1, New Mathematical Monographs, Cambridge University Press, 2016.

[29] T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. IHÉS **117** (2013), 271–328; arXiv:1111.3209.

[30] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[31] J. Milnor and J. D. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies **76**, Princeton University Press, 1974.

[32] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990), 393–429.

[33] D. S. Freed, *Anomalies and invertible field theories*, arXiv:1404.7224 (2014).

[34] D. S. Freed and M. J. Hopkins, *Reflection positivity and invertible topological phases*, arXiv:1604.06527 (2016).

[35] A. Yu. Kitaev, *Fault-tolerant quantum computation by anyons*, arXiv:quant-ph/9707021 (1997); Ann. Phys. **303** (2003), 2–30.

[36] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, *Topological quantum memory*, J. Math. Phys. **43** (2002), 4452–4505; arXiv:quant-ph/0110143.

[37] D. Gottesman, *Stabilizer codes and quantum error correction*, Caltech PhD thesis, arXiv:quant-ph/9705052 (1997).